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# STABILITY AND BIFURCATION BEHAVIOR OF AN INVERTED PENDULUM WITH FOLLOWER FORCE

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***Abstract:*** *The study of dynamics of an inverted pendulum with follower force is of interest to several fields of physics, mechanics and engineering. Using Lyapunov-Andronov's theory, we find a new analytical formula for first Lyapunov value at the boundary of stability. It enables one to study in details the bifurcation behavior of dynamic systems of the above type. We check the validity of our analytical results on the first Lypunov's value by numerical simulations. Our numerical analysis suggests that follower force have stabilization dynamical role and hard stability loss take place.*

## 1. Introduction

Dynamical systems are the study of the long-term evolution of evolving deterministic systems. Evolution is function which describe the state of a system as a function of time and which satisfy the equation(s) of motion of the system. A system whose time evolution equations appear in a nonlinear form is termed nonlinear. It is widely known that almost all mechanical systems are nonlinear. A dynamical system may be defined as a deterministic mathematical prescription for evolving the state of a system forward in time (continuous variable or discrete integer-valued variable) [16].

It is well-known that the simplest type of evolution is stationary, where the state is constant in time, i.e. the qualitative structure of the dynamical system flow does not change for sufficiently small variations of the parameters(s). Next we also know periodic evolutions, where after a fixed period, the system always returns to the same state. Stationary and periodic evolutions are regular and predictable. In some dynamical systems one meets evolutions (behaviors) that are not so regular and predictable. In these cases (where the unpredictability can be established), we speak of chaotic behavior [1].

The concept of stability is based on dynamical Lyapunov's definition which apply to equilibrium states as well as motions. The term 'dynamical stability' is somewhat ambiguous and on the other hand the term dynamical instability is often used to indicate dynamical bifurcations from an equilibrium state (local bifurcations) or global bifurcations.

The study of possible changes in the structure of the orbits (evolution) of a dynamical system as parameters are varied is called bifurcation theory. A parameter value for which the flow does not have stable orbit structure is called a bifurcation value, and the system is said to be at a bifurcation point. Thus, a bifurcation is a change of the topological type of the system as its parameters pass through a bifurcation (critical) value.

From an analytical point of view, the Andronov-Hopf bifurcation leads to the appearance from the equilibrium state, of small-amplitude periodic oscillations. This local bifurcation is the simplest one, i.e. it can be detected if fix any small neighborhood of the equilibrium, and can be types: (i) supercritical (soft loss of stability) and (ii) subcritical (hard loss of stability). In the scientific literature the local bifurcations are refered as bifurcations of equilibria or fixed points, although that we analyze not these points but the whole phase portraits near the equilibria.

There also global bifurcations, which cannot be detected by looking at small neighborhoods of fixed (equilibrium) points or cycles.

The study of dynamics of an inverted pendulum with follower force has a long history. The earliest theoretical investigation is that by Pfluger [2] who studied the effect of follower force on the dynamical behavior of an elastic bar. He used Euler's concept to find the critical magnitude of the follower force that causes in stability of the rectilinear equilibrium mode. It is interesting that Pfluger obtained an unexpected conclusion (which is contradicted experimental observations) for missing critical load. Later, Ziegler [3-5] resolved Pfluger paradox by modeling an elastic cylindrical pipe with a moving fluid inside by a double simple pendulum with a follower force. This follower force is appeared to be responsible for the disagreement between Pfluger's theoretical results and experimental observations.

In [6], Hagedorn examined a mathematical model with two degrees of freedom previously studied by Zeigler, but Zeigler's linear damping is replaced by a nonlinear term, as is usual for structural damping. It is obtained there that nonlinear damping gives rise to a behavior similar to that due to linear damping; however the stability of zero position depends only on the ratio of the two damping coefficients. Using some ideas of Zeigler, in [7] the structure of the dissipative operators are studied in both case for classical dissipative forces-as defective or ideal according to whether they do or d not exceed a critical parameter. Thus, the necessary conditions are established for perturbations effected by small linear forces. Some additional results are found by Kounadis [8] using a nonlinear analysis, i.e. it is deduced that the critical states corresponding to both types of instability may become unstable if a slight material nonlinearity is included; then, the mechanism of divergence and flutter instability change from stable to unstable and vice versa for a critical value of the material nonlinearity which depends on the non-conservativeness loading parameter.

In the recent years, there has been particular interest in the study of nonlinear behavior of inverted pendulums with or not follower force [10-13]. It was obtained that two limit cycles (a stable $L^+$ and an unstable $L^-$) are born for a critical value of the follower force. Also, when the magnitude of the follower force decreases, the limit cycles move toward each other.

In this paper we are interested in the stability and bifurcation behavior of double inverted pendulum with follower force. Here (following [13]), the differential equations of motion of the inverted pendulum have the form

$$
(1)\quad \left|\begin{aligned}
&(m_1+m_2)l_1^2\ddot{\varphi}_1+m_2l_1l_2\ddot{\varphi}_2\cos(\varphi_1-\varphi_2)+m_2l_1l_2\dot{\varphi}_2^2\sin(\varphi_1-\varphi_2)-(m_1+m_2)gl_1\sin\varphi_1+\\
&\quad +cl_1(l_1\sin\varphi_1+l_2\sin\varphi_2)\cos\varphi_1+(c_1+c_2)\varphi_1-c_2\varphi_2+(\mu_1+\mu_2)\dot{\varphi}_1-\mu_2\dot{\varphi}_2-F_fl_1\sin(\varphi_1-\varphi_2)=0,\\
&m_2l_1l_2\ddot{\varphi}_1\cos(\varphi_1-\varphi_2)+m_2l_2^2\ddot{\varphi}_2-m_2l_1l_2\dot{\varphi}_1^2\sin(\varphi_1-\varphi_2)-m_2gl_2\sin\varphi_2+\\
&\quad +cl_2(l_1\sin\varphi_1+l_2\sin\varphi_2)\cos\varphi_2-c_2(\varphi_1-\varphi_2)-\mu_2\left(\dot{\varphi}_1-\dot{\varphi}_2\right)=0,
\end{aligned}\right.
$$

where $\varphi_1$ and $\varphi_2$ are the generalized coordinates (the angular displacements measured from the downward vertical) of the motion of double inverted pendulum (see Fig. 1), $m_1$ and $m_2$ are the masses of material point $P_1$ and $P_2$, $OP_1 = l_1$ and $P_1P_2 = l_2$ are the imponderable links, *O* and *A* are the elastic and linear viscous joints, $\mu_1$ and $\mu_2$ are the viscosity coefficients representing the external friction in the lower joint *O* and in the intermediate joint *A*, *c* is the stiffness at the upper and of the pendulum, $c_1$ and $c_2$ are the stiffness of the spiral springs in the joints *O* and *A*, and $\vec{F}_f$ is the follower force, respectively.

If the angles $\varphi_1$ and $\varphi_2$ are small (i.e. $\varphi_1 \approx 0^0 - 6^0$ and $\varphi_2 \approx 0^0 - 6^0$), (1) can be linerized by let $\sin\varphi_1 = \varphi_1, \cos\varphi_1 = 1, \sin\varphi_2 = \varphi_2, \cos\varphi_2 = 1, \sin(\varphi_1 - \varphi_2) = \varphi_1 - \varphi_2$ and $\cos(\varphi_1 - \varphi_2) = 1$. After substitution of the previous equalities into (1) and accomplishing some transformations, the system (1) takes the form

$$
(2)\quad \left|\begin{aligned}
&\ddot{\varphi}_1=-a_1\dot{\varphi}_1+a_2\dot{\varphi}_2+a_3\varphi_1+a_4\varphi_2-a_5\varphi_1\dot{\varphi}_1+a_5\varphi_2\dot{\varphi}_1^2,\\
&\ddot{\varphi}_2=a_6\dot{\varphi}_1-a_7\dot{\varphi}_2-a_8\varphi_1-a_9\varphi_2+a_{10}\varphi_1\dot{\varphi}_1^2-a_{10}\varphi_2\dot{\varphi}_1^2,
\end{aligned}\right.
$$

where

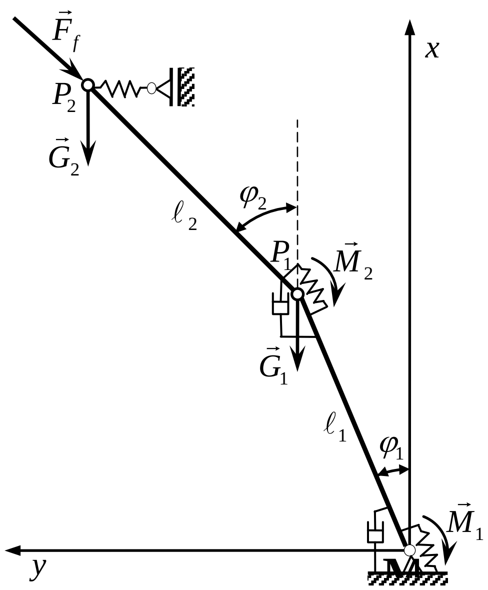


**Figure 1.** Schematic diagram of the motion of a double inverted pendulum with follower force.

$$a_1 = \frac{1}{m_1 l_1}\left(\frac{\mu_2}{l_2} - \frac{\mu_1 + \mu_2}{l_1}\right), \quad a_2 = \frac{\mu_2}{m_1 l_1}\left(\frac{1}{l_1} + \frac{1}{l_2}\right), \quad a_3 = \frac{1}{m_1 l_1}\left[F_f + \frac{c_1 + c_2}{l_1} + (m_1 + m_2)g - \frac{c_2}{l_2}\right],$$

(3)
$$a_4 = \frac{1}{m_1 l_1}\left(\frac{c_2}{l_1} + \frac{c_2}{l_2} - m_2 g - F_f\right), \quad a_5 = \frac{m_2}{m_1}\left(1 + \frac{l_2}{l_1}\right), \quad a_6 = \frac{1}{l_2}\left[\frac{1}{m_1}\left(\frac{\mu_2}{l_2} - \frac{\mu_1 + \mu_2}{l_1}\right) + \frac{\mu_2}{m_2 l_2}\right],$$

$$a_7 = \frac{\mu_2}{l_2}\left[\frac{1}{m_2 l_2} + \frac{1}{m_1}\left(\frac{1}{l_1} + \frac{1}{l_2}\right)\right], \quad a_8 = \frac{1}{l_2}\left\{\frac{1}{m_1}\left[F_f + \frac{c_1 + c_2}{l_1} + (m_1 + m_2)g - \frac{c_2}{l_2}\right] + \frac{c l_1}{m_2} - \frac{c_2}{m_2 l_2}\right\},$$

$$a_9 = \frac{1}{l_2}\left[\frac{1}{m_1}\left(\frac{c_2}{l_2} - m_2 g + \frac{c_2}{l_1} - F_f\right) + \frac{c l_2}{m_2} + \frac{c_2}{m_2 l_2} - g\right], \quad a_{10} = \frac{l_1}{l_2}\left[1 + \frac{m_2}{m_1}\left(1 + \frac{l_2}{l_1}\right)\right].$$

The aim of this article is to elucidate how the dynamics of the system (2) is controlled by the follower force $\vec{F}_f$ and some other critical model parameters. Thus, the plan of the paper is as follows: in Section 2 a qualitative (bifurcation) analysis is performed. In Section 3 we explore numerically the model. Finally, in Section 4, we discuss and unify results from previous sections.

## 2. Qualitative analysis

In this section, we shall consider the system (2) which present a mathematical model of double inverted pendulum with follower force. Let us denote

(4)
$$z_1 = \varphi_1,\ z_2 = \dot{\varphi}_1,\ z_3 = \varphi_2,\ z_4 = \dot{\varphi}_2 .$$

After substitution of (4) into (2) and accomplishing some transformations, the system (2) is reduced to the four first-order differential equations:

(5)
$$\begin{aligned}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= a_3 z_1 - a_1 z_2 + a_4 z_3 + a_2 z_4 - a_5 z_1 z_2^2 + a_5 z_3 z_2^2, \\
\dot{z}_3 &= z_4, \\
\dot{z}_4 &= -a_8 z_1 + a_6 z_2 - a_9 z_3 - a_7 z_4 + a_{10} z_1 z_2^2 - a_{10} z_3 z_2^2,
\end{aligned}$$

The steady (fixed points in the phase space) states of the system (5), $E = \left(\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4\right)$, are founded by equating the right-hand sides of (5) to zero, i.e.

(6)
$$\bar{z}_2 = \bar{z}_4 = 0, \quad \bar{z}_1 = -\frac{a_4}{a_3}\bar{z}_3, \quad \bar{z}_3\left(\frac{a_4 a_8}{a_3} - a_9\right) = 0.$$

It is seen that system (1) has only one fixed point in the zero, i.e.

(7)
$$\bar{z}_1 = \bar{z}_2 = \bar{z}_3 = \bar{z}_4 = 0,$$

or many fixed points different from zero for $\bar{z}_1$ and $\bar{z}_3$ if $\frac{a_4 a_8}{a_3} - a_9 = 0$. From dynamical point of view, the system possesses many additional stationary points when the follower force $F_f$ is equal to

(8)

$$F_f = \frac{c_2(2c_2l_1 - c_1l_2 - Al_2) + cl_2\{[m_2l_1 + l_2(m_1 + m_2)]gl_1l_2 + (c_1 + c_2)l_2^2 - c_2l_1(2l_2 + l_1)\} - m_2gl_2^2(c_1 + c_2 + A)}{B}$$

where

(9) $$A = (m_1 + m_2)gl_1, \quad B = l_1l_2[2c_2 + m_2l_2g - cl_2(l_1 + l_2)].$$

It is seen that $B$ is different from zero if the following condition is valid

(10) $$c \neq \frac{2c_2 + m_2l_2g}{l_2(l_1 + l_2)}.$$

Thus, a natural question that arises from the above study is, which case is the best from the inverted pendulum stability point of view? Of course, this is the first case i.e. when the equality (8) is not valid. Also the equilibrium state (7) must be stabilized.

In order to investigate the character of the fixed point (7), following [14] the Routh-Hurwitz conditions for stability in this case can be written in the form

(11) $$p = a_1 + a_7 > 0,$$

(12) $$q = a_9 + a_1a_7 - a_3 - a_2a_6 > 0,$$

(13) $$r = a_2a_8 + a_1a_9 - a_3a_7 - a_4a_6 > 0,$$

(14) $$s = a_4a_8 - a_3a_9 > 0,$$

(15) $$R = pqr - sp^2 - r^2 > 0.$$

For sufficiently large values of the follower force $F_f$ (different from these in (8)) and some other model parameters, the condition (15) can be broken. In this case the steady state (7) becomes unstable, i.e. the well-known phenomenon “loss of stability” takes place. In terms of the dynamical system theory there exist “soft” and “hard” loss of stability, i.e. the stability boundaries of equilibrium states are safe and dangerous [14, 15]. Safe boundaries are such that crossing over them leads to small quantitative changes of the system’s state and reversible behaviour take place. Opposite, dangerous boundaries are such that very small perturbations of the system lead to significant and irreversible changes in her behaviour. In order to define whether the corresponding boundary of stability is safe or dangerous, it is necessary to calculate the so-called first Lyapunov value (coefficient) $L_1(\lambda_0)$ on the stability boundary $R = 0$. Here we note that when $L_1(\lambda_0) < 0$ the boundary of stability is safe, and when $L_1(\lambda_0) > 0$ -dangerous. In case of fourt first-order differential equations, this value can be determined analytically by using the formula in [14]. In our case, after accomplishing some transformations and algebraic operations for the first Lyapunov value, we obtain the following form

(16) $$L_1(\lambda_0) = \frac{3\pi}{4b\Delta_0}\{S_4[3\alpha_{21}^2S_3 + \alpha_{22}(2\alpha_{21}S_2 + \alpha_{22}S_3)] + S_1[\alpha_{21}(\alpha_{21}S_2 + 2\alpha_{22}S_3) + 3\alpha_{22}^2S_2]\},$$

where

(17) $$S_1 = \alpha_{24}^{'}a_{10} - \alpha_{22}^{'}a_5, \quad S_2 = \alpha_{12} - \alpha_{32}, \quad S_3 = \alpha_{11} - \alpha_{31}, \quad S_4 = \alpha_{14}^{'}a_{10} - \alpha_{12}^{'}a_5.$$

Here, we note that

(18)
$$b^2 = \frac{r}{p}, \quad \alpha_{12}^{'} = -\begin{vmatrix} \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{42} & \alpha_{43} & \alpha_{44} \end{vmatrix}, \quad \alpha_{14}^{'} = -\begin{vmatrix} \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{32} & \alpha_{33} & \alpha_{34} \end{vmatrix},$$
$$\alpha_{22}^{'} = \begin{vmatrix} \alpha_{11} & \alpha_{13} & \alpha_{14} \\ \alpha_{31} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{43} & \alpha_{44} \end{vmatrix}, \quad \alpha_{24}^{'} = \begin{vmatrix} \alpha_{11} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{33} & \alpha_{34} \end{vmatrix}, \quad \Delta_0 = \det\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{vmatrix},$$

and

(19)
$$\begin{gathered}
\alpha_{11} = a_4, \quad \alpha_{12} = -ba_2, \quad \alpha_{21} = -b^2 a_2, \quad \alpha_{22} = -ba_4, \quad \alpha_{31} = -a_3 - b^2, \quad \alpha_{32} = -ba_1, \\
\alpha_{41} = -b^2 a_1, \quad \alpha_{42} = b\left(b^2 + a_3\right), \quad \alpha_{23} = a_3\left[m\left(m + a_7\right) + a_9 - n^2\right] - a_8\left(a_4 + ma_2\right), \\
\alpha_{24} = -n\left[a_3\left(2m + a_7\right) - a_2 a_8\right], \quad \alpha_{33} = a_3 a_6 - a_8\left(a_1 + m\right), \quad \alpha_{14} = n\left[n^2 - \left(a_1 + m\right)\left(2m + a_7\right) + a_2 a_6\right], \\
\alpha_{43} = a_8\left[n^2 - m\left(a_1 + m\right)\right] + ma_3 a_6, \quad \alpha_{44} = n\left[a_8\left(2m + a_1\right) - a_3 a_6\right], \quad \alpha_{34} = na_8, \\
\alpha_{13} = -n^2\left(a_1 + a_7 + 3m\right) + \left(a_1 + m\right)\left[m\left(m + a_7\right) + a_9\right] - a_6\left(a_4 + ma_2\right), \quad m = -\frac{p}{2}, \\
n^2 = \frac{sp}{r} - \frac{p^2}{4}\left(n > 0\right), \quad n = \sqrt{-\Delta_1^*}, \quad \Delta_1^* = \frac{sp}{r} - \frac{p^2}{4}.
\end{gathered}$$

As we can see into (16), the value of the first Lyapunov value on the boundary of stability $R = 0$ is negative/positive in the computed bifurcation points, which means that the boundary of stability is safe/dangerous and therefore 'soft' (reversible)/'hard' (nonreversible) stability loss takes place.

In the following section we will calculate numerically the value of $L_1\left(\lambda_0\right)$ (using (16)) and demonstrate the behaviour of the model (5) which is equivalent to system (2).

## 3. Numerical analysis

Here we numerically calculate the value of the first Lyapunov value on the boundary of stability $R = 0$. After that, we numerically illustrate the stability and existence of periodic solutions via Andronov-Hopf bifurcation (supercritical/subcritical) in model (2). Some of the corresponding numerical values of the model parameters are taken from other reports [11-13] in the form

(20)
$$\begin{gathered}
m_1 = 10\,[kg], \quad m_2 = 5\,[kg], \quad l_1 = 1.61\,[m], \quad l_2 = 0.89\,[m], \\
c_1 = c_2 = 600\,[Nm], \quad \mu_1 = \mu_2 = 10\,[Nms], \quad c = 900\,[N/m].
\end{gathered}$$

Our model include one additional parameter, $F_f$ $[N]$, which we assume to vary in the following interval $F_f \in [220, 240]\,[N]$. It is important to note here that all values of follower force from this interval are different from critical one (calculated in (8)), i.e. the system (5) has only one fixed point. The initial conditions of all variables in our model are $z_1 = z_2 = z_4 = 0$ and $z_3 = 0.1$.

In order to compare the predictions with numerical results, the governing equations of model (2) were solved numerically using Matlab [17]. In Figure 2, the stable solutions for the generalized coordinates (the angular displacements measured from the downward vertical) and velocities of the motion of double inverted pendulum are shown for $F_f = 235$. It is evident that after several physically accepted fluctuations $z_1, z_2, z_3$ and $z_4$ approach constant values (equilibrium state). In other words, in this case the conditions (11)-(15) are satisfied and the steady state of system (5) is locally asymptotically stable.

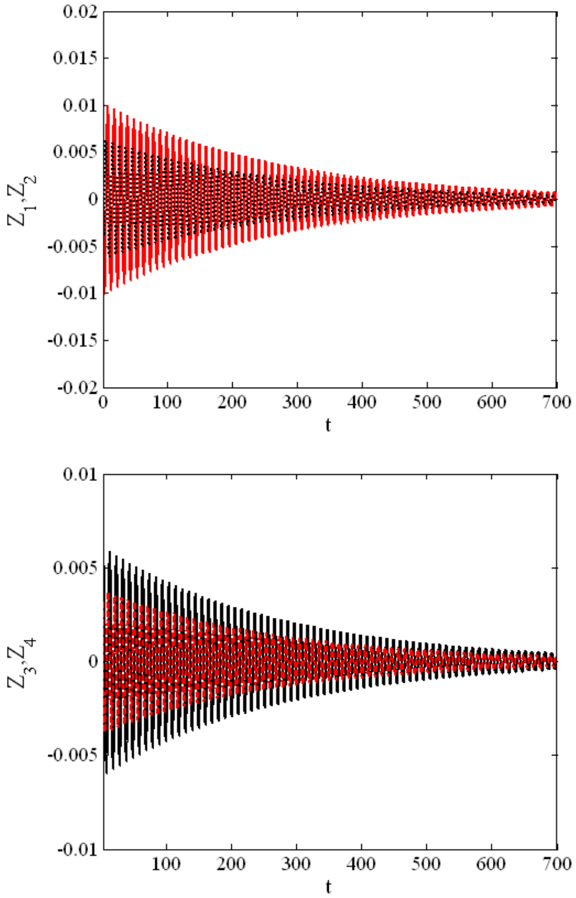


**Figure 2.** Stability solution of system (5) at $F_f = 235$. All other model parameters are those from (20). The time (t) is in seconds. Here we note that $z_2$ and $z_4$ are dashed lines.

Figure 3 depicts the case when the follower force $F_f = 227.4122$. The left panel demonstrates the time behavior of the generalized coordinate $\varphi_1(z_1)$ and the right panel of generalized coordinate $\varphi_2(z_3)$. It is seen that the system has periodic oscillations with period one. In this case, the system is on boundary of stability $R = 0$. In the theory of dynamic system, there the system is structurally unstable. Using (16), we calculate the first Lypunov value on boundary $R = 0$. Hence, we obtain that $L_1 = 210.34 > 0$, i.e. the boundary of stability $R = 0$ is dangerous. In the case of transition through this boundary from positive values to negative, an unstable limit cycle emerges- hard loss of stability. Inversely, in the case of transition from negative to positive, the unstable limit cycle disappears. In the following Figure 4 this type (hard loss of stability) is numerically demonstrated.

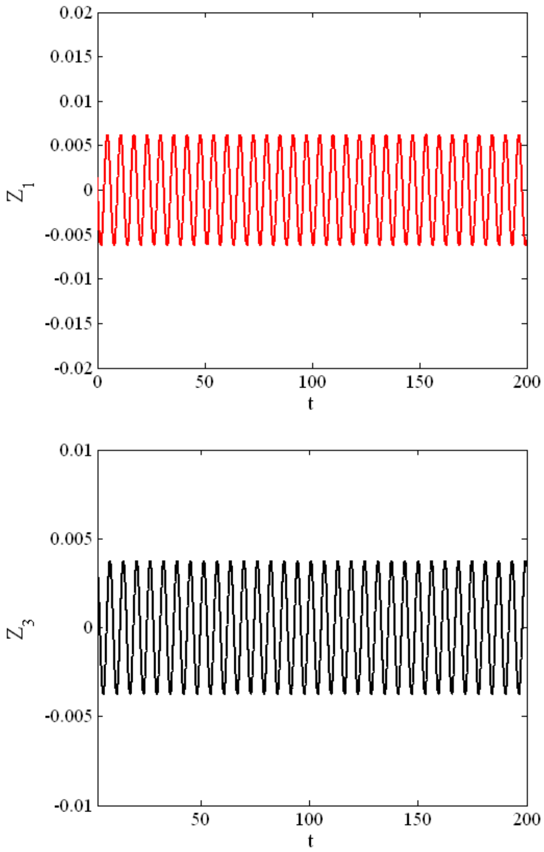


**Figure 3.** Solution of system (5) on the boundary of stability R=0, when $F_f = 227.4122$. The time (t) is in seconds.

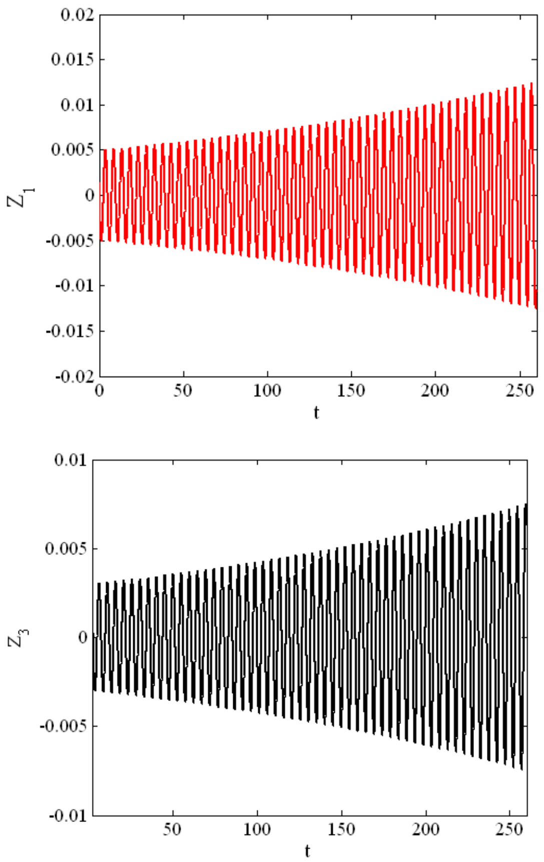


**Figure 4.** Hard loss of stability of system (5) for $F_f = 220$, i.e. an unstable limit cycle emerges.

## 4. Conclusions

In the present study, using Lyapunov-Andronov's theory, the problem of stability and bifurcation behavior of an inverted pendulum with follower force was investigated. The model resulted in four nonlinear ODEs.

The basic view that the follower force is a key factor in the dynamic behavior of the system was confirmed by the analytical calculations and numerical simulations. From the viewpoint of the qualitative theory of ODEs, follower force, $F_f$, appears as a bifurcation parameter on whose values the altered (stable or unstable) behavior of the model depends. For follower forces smaller than bifurcation one, the steady state is unstable and an unstable limit cycle emerges. In contrast, a follower force bigger than bifurcation one would provoke damped oscillations around a stable steady state. We can say that in this situation follower forces have a stabilizing role. From a dynamical perspective, the loss of stability is hard (the boundary of stability is dangerous) and might be related to emergence of new configurations in the inverted state.

Finally, we find a new analytical formula for the first Lyapunov value at the stability limit. It enables one to study in detail (in a further study) the bifurcation behavior of system (2) for other numerical values of the system's parameters.

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# УСТОЙЧИВОСТ И БИФУРКАЦИОННО ПОВЕДЕНИЕ НА ОБЪРНАТО МАХАЛО С ПРОСЛЕДЯВАЩА СИЛА

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***Ключови думи:*** *обърнато махало, проследяваща сила, устойчивост, бифуркационно поведение*

***Резюме:****Изучаването на динамиката на обърнато махало с проследяваща сила представлява интерес за редица научни области като физика, механика и др. Използвайки теорията на Ляпунов-Андронов ние намираме нова аналитична формула за първата Ляпунова величина на границата на устойчивост. Това дава възможност да бъде изучено в детайли бифуркационното поведение на динамичната система от споменатия по-горе вид. Проверката на верността на получените от нас аналитични резултати става с помощта на числени симулации. Направеният числен анализ показва, че проследяващата сила има стабилизираща динамична роля, както и това че може да се появи твърда (необратима) загуба на устойчивост.*