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ON THE DYNAMICS OF AN ELASTIC MATHEMATICAL PENDULUM WITH A MOVABLE SUSPENSION POINT

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Abstract: *The present paper treats an elastic mathematical pendulum with a movable suspension point. The load, which is taken as a point mass, is suspended with a homogeneous, elastic, massless string. The suspension point is fixed at center of a homogeneous disc, which rolls without slipping along a horizontal plane. The disc models a mechanism for moving in a stationary regime of motion. Non-linear mechanics is used to determine the law of motion of the elastic mathematical pendulum and the dynamic loading of the flexible cable.*

1. Introduction

Mechanical systems with flexible coupling, and particularly with suspended weights, are widely used to automate the key technological, subsidiary, and transport processes in industry. The increase of productivity and in turn of the working speeds leads to an increased dynamical loading on the elements of these mechanical systems, which requires a thorough analysis. Modeling of operation of the mechanical systems with suspended weights can be done by a mathematical pendulum with a movable suspension point. Pendulums under various assumptions and in various configurations have been subject of many studies and continuing into current times. Interest in this problem stems the fact that the elastic pendulum is a very rich dynamical system and it can serve as a model for many engineering problems. Among the numerous studies used numerical methods we note [1,2,3,4,5,6,7]. Analytical methods to solve the differential equations, which describe the systems` motion, were used in [8,9,10,11,12,13,14,15,16,17,18].

The aim of this study is to use analytical methods to investigate with higher accuracy then [16] the dynamics of mechanical systems with suspended weights, including mechanisms used for moving and hoisting, which can be modeled as an elastic mathematical pendulum with a movable suspension point.

2. Mechanic-mathematical model

The mechanical systems with suspended weights, including mechanisms used for moving and hoisting, can be modeled as a homogeneous disk, which rolls without slipping along a horizontal plane, and its center is the suspension point of an elastic mathematical pendulum (Fig. 1).

The dynamic model is based on the following assumptions. The mechanical system is discrete. The disc 1 is taken as a homogeneous disc with mass m_1, radius R, and a geometric center C, where a load 3 taken as point mass m_3 is attached. The angle of rotation of the disk is φ. The load 2, taken as a point mass m_2, is attached to an elastic, massless string with a spring rate c, which is suspended from a pivot at the geometric center of the disc. The load is moving in the vertical plane. The length of the string at an elastic equilibrium is l_{st}, at an arbitrary point in time - ρ, the angle of deviation of the string CA from the vertical axis is θ. The rotational moment of the motor T_M, which is applied on the disc, can be described with the following formula**:**

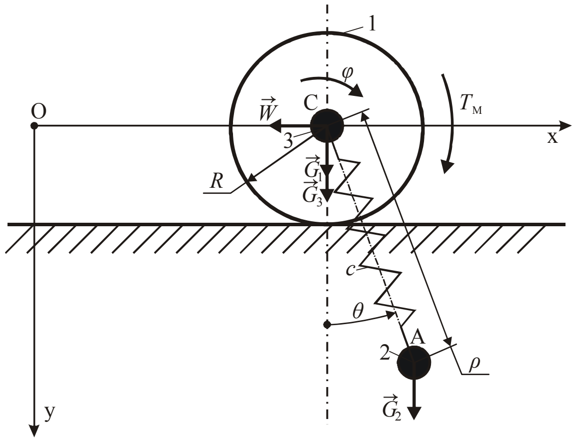

Fig. 1 Dynamic Model

(2.1) $$T_M = a - b\dot{\varphi}$$

which approximates the stable linear portion of the mechanical characteristic curve of the asynchronous electric motor during a stationary regime of motion, where *a* and *b* are constants. The friction forces of motion are denoted with W and include the friction of traveling along a horizontal path, the friction of traveling along a slope, the air drag, etc.

The dynamic model has three degrees of freedom. The following are generalized coordinates: φ, ρ, θ. The system can be described using three Lagrange equations of the second kind:

(2.2) $$\frac{d}{dx}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad , \quad (q_1 = \varphi, q_2 = \theta, q_3 = \rho).$$

Using the aforementioned assumptions and notation, and letting $M = \frac{3}{2}m_1 + m_2 + m_3$, we get the following result for the kinetic energy T:

(2.3) $$T = \frac{1}{2}M\,R^2\dot{\varphi}^2 + m_2R\dot{\varphi}\dot{\rho}\sin\theta + m_2R\rho\dot{\varphi}\dot{\theta}\cos\theta + \frac{1}{2}m_2\dot{\rho}^2 + \frac{1}{2}m_2\rho^2\dot{\theta}^2 .$$

The mechanical system is non-conservative, which means that the generalized forces Q_j can be determined using the formula:

(2.4) $$Q_j = -\frac{\partial \Pi}{\partial q_j} + \tilde{Q}_j \qquad (j = 1,2,3).$$

We get the following result for the potential energy Π and the generalized non-potential forces $\tilde{Q}_j$:

(2.5) $$\Pi = \frac{1}{2}c\left(\rho - l_{st} + \frac{m_2 g}{c}\right) - m_2 g\rho\cos\theta,\ \tilde{Q}_1 = T_M - WR\ ,\ \tilde{Q}_2 = 0,\ \tilde{Q}_3 = 0 .$$

Substituting (2.3)-(2.5) into (2.2), we obtain the following system of three nonlinear differential equations, describing the motion of the mechanical system with suspended load:

$$
\begin{aligned}
&M R^2\ddot{\varphi}+m_2R\ddot{\rho}\sin\theta+2m_2R\dot{\rho}\dot{\theta}\cos\theta+m_2R\rho\ddot{\theta}\cos\theta-m_2R\rho\dot{\theta}^2\sin\theta=a-b\dot{\varphi}-W R,\\
(2.6)\qquad &m_2R\rho\ddot{\varphi}\cos\theta+m_2\rho^2\ddot{\theta}+2m_2\rho\dot{\rho}\dot{\theta}=-m_2 g\rho\sin\theta,\\
&m_2R\ddot{\varphi}\sin\theta+m_2\ddot{\rho}-m_2\rho\dot{\theta}^2=m_2g\cos\theta-c(\rho-l_0).
\end{aligned}
$$

3. Analytical solution method

The choice of the type of the functions in $\varphi(t),\theta(t),\rho(t)$ is based on the expected motion of the mechanical system. We assume that the angular velocity of the disk ω oscillate around the stationary angular velocity ω_{st}, and the change in length and the deviation of the string happens from a given position, i.e.:

(3.1) $$\varphi(t)=\omega_{st}t+\mu'\psi'(t),\quad \theta(t)=\theta^*+\mu'\eta'(t),\quad \rho(t)=\rho^*+\mu'\xi'(t),$$

where μ' is a small positive parameter.

Substituting (3.1) into (2.6), and considering that $\sin\left(\theta^*+\mu'\eta'\right)\approx\sin\theta^*+\mu'\eta'\cos\theta^*$ and $\cos\left(\theta^*+\mu'\eta'\right)\approx\cos\theta^*-\mu'\eta'\sin\theta^*$, we get the following form for the first equation from (2.6):

$$
\begin{aligned}
&M R^2\mu'\ddot{\psi}'+m_2R\,\mu'\ddot{\xi}'\left(\sin\theta^*+\mu'\eta'\cos\theta^*\right)+2m_2R\mu'\dot{\xi}'\mu'\dot{\eta}'\left(\cos\theta^*-\mu'\eta'\sin\theta^*\right)+\\
(3.2)\qquad &+m_2R(\rho*+\mu'\xi')\mu\ddot{\eta}'\left(\cos\theta^*-\mu'\eta'\sin\theta^*\right)-\\
&-m_2R(\rho*+\mu'\xi')\mu'^2\dot{\eta}'^2\left(\sin\theta^*+\mu'\eta'\cos\theta^*\right)=a-b(\omega_{st}+\mu'\dot{\psi}')-WR.
\end{aligned}
$$

We can write for the coefficients of μ'^0:

(3.3) $$a-b\omega_{st}-WR=0\Rightarrow\omega_{st}=\frac{a-WR}{b}.$$

The second equation from (2.6) can be obtained in the following form:

$$
\begin{aligned}
&m_2R\left(\rho^*+\mu'\xi'\right)\mu'\ddot{\psi}'\left(\cos\theta^*-\mu'\eta'\sin\theta^*\right)+m_2\left(\rho^*+\mu'\xi'\right)^2\mu'\ddot{\eta}'+\\
(3.4)\qquad &+2m_2\left(\rho^*+\mu'\xi'\right)\mu'\dot{\xi}'\mu'\dot{\eta}'=-m_2g\left(\rho^*+\mu'\xi'\right)\left(\sin\theta^*+\mu'\eta'\cos\theta^*\right),
\end{aligned}
$$

and setting the coefficients of μ'^0 to zero: $m_2g\rho^*\sin\theta^*=0$. As a result we obtain:

(3.5) $$\sin\theta^*=0\Rightarrow\theta^*=0,$$

because $m_2\neq0$, $\rho^*\neq0$ and $g=9{,}81\,\mathrm{m/s^2}$.

We obtain the third equation from (2.6) in the following form:

$$
\begin{aligned}
&m_2R\mu'\ddot{\psi}'\left(\sin\theta^*+\mu'\eta'\cos\theta^*\right)+m_2\mu'\ddot{\xi}'-m_2\left(\rho^*+\mu'\xi'\right)\left(\mu'\dot{\eta}'\right)^2=\\
(3.6)\qquad &=m_2g\left(\cos\theta^*-\mu'\eta'\sin\theta^*\right)-c\left(\rho^*+\mu'\xi'-l_0\right),
\end{aligned}
$$

and setting the coefficients of μ'^0 to zero: $m_2g\cos\theta^*-c\rho^*+cl_0=0$, we obtain:

(3.7) $$\rho^*=\ell_0+\frac{m_2g}{c}=l_{st},$$

where l_0 denotes the free length of the string and $\cos\theta^*=1$ from (3.5).

We seek the functions $\varphi(t),\theta(t),\rho(t)$, using (3.1), (3.3), (3.5), (3.7), in the form of power series of the small parameter μ:

(3.8)
$$\varphi(t)=\psi(t)=\psi_0(t)+\mu\,\psi_1(t), \qquad \theta(t)=\eta(t)=\eta_0(t)+\mu\,\eta_1(t),$$
$$\rho(t)=l_{st}+\xi(t)=l_{st}+\xi_0(t)+\mu\,\xi_1(t).$$

Substituting (3.8) into (2.6), and assuming that the deviations of the string are small, i.e. $\sin\theta\approx\theta$, $\cos\theta\approx 1$, we can write (2.6) in the following form:

(3.9)
$$J_0\ddot{\psi}+B_0\dot{\psi}=C_0+\mu\left(a_{11}\eta\ddot{\xi}+a_{12}\ddot{\eta}\xi+a_{13}\dot{\eta}\dot{\xi}+a_{14}\ddot{\eta}+a_{15}\eta\dot{\eta}^2+a_{16}\eta\dot{\eta}^2\xi\right),$$
$$\ddot{\eta}+k_2^{\,2}\eta=\mu\left(a_{21}\ddot{\psi}+a_{22}\ddot{\eta}\xi+a_{23}\dot{\eta}\dot{\xi}\right),$$
$$\ddot{\xi}+k_3^{\,2}\xi=\mu\left(a_{31}\dot{\eta}^2+a_{32}\eta\ddot{\psi}+a_{33}\xi\dot{\eta}^2\right),$$

where we let: $J_0=MR^2$, $B_0=b$, $C_0=a-WR$, $a_{11}=-m_2^{\,2}R$, $a_{12}=-m_2^{\,2}R$, $a_{13}=-2m_2^{\,2}R$, $a_{14}=-m_2^{\,2}Rl_{st}$, $a_{15}=m_2^{\,2}Rl_{st}$, $a_{16}=m_2^{\,2}R$, $a_{21}=-\dfrac{Rm_2}{l_{st}}$, $a_{22}=-\dfrac{m_2}{l_{st}}$, $a_{23}=-2\dfrac{m_2}{l_{st}}$, $a_{31}=m_2l_{st}$, $a_{32}=-m_2R$, $a_{33}=m_2$, $\mu=\dfrac{1}{m_2}(m_2>1)$ - a small positive parameter, $k_2=\sqrt{\dfrac{g}{l_{st}}}$, $k_3=\sqrt{\dfrac{c}{m_2}}$ - natural frequency of the mathematical pendulum and the spring-load system.

We seek the solution of the first equation from (3.9) by means of the method of the small parameter accurate to the first power of μ, and the second and third equation applying the Lyapunov-Lindstedt method. We expand the square of the natural frequency in power series about the powers of μ, i.e.

(3.10)
$$p_2^{\,2}=k_2^{\,2}+h_1\mu \quad , \quad p_3^{\,2}=k_3^{\,2}+r_1\mu,$$

Substituting the right side of (3.8) and (3.10) into (3.9) and comparing the coefficients of the equal powers of μ, we get the following system of equations, which can be used to determine the functions $\psi_0(t),\psi_1(t),\eta_0(t),\dots$ and the constants h_1, r_1:

(3.11)
$$J_0\ddot{\psi}_0+B_0\dot{\psi}_0=C_0,$$
$$J_0\ddot{\psi}_1+B_0\dot{\psi}_1=a_{11}\eta_0\ddot{\xi}_0+a_{12}\ddot{\eta}_0\xi_0+a_{13}\dot{\eta}_0\dot{\xi}_0+a_{14}\ddot{\eta}_0+a_{15}\eta_0\dot{\eta}_0^{\,2}+a_{16}\eta_0\dot{\eta}_0^{\,2}\xi_0,$$

(3.12)
$$\ddot{\eta}_0+p_2^{\,2}\eta_0=0,$$
$$\ddot{\eta}_1+p_2^{\,2}\eta_1=h_1\eta_0+a_{21}\ddot{\psi}_0+a_{22}\ddot{\eta}_0\xi_0+a_{23}\dot{\eta}_0\dot{\xi}_0,$$

(3.13)
$$\ddot{\xi}_0+p_3^{\,2}\xi_0=0,$$
$$\ddot{\xi}_1+p_3^{\,2}\xi_1=r_1\xi_0+a_{31}\dot{\eta}_0^{\,2}+a_{32}\eta_0\ddot{\psi}_0+a_{33}\xi_0\dot{\eta}_0^{\,2}.$$

We seek the solution of (3.9) using the following initial conditions:

(3.14)
$$t=0,\ \varphi(0)=\varphi_0,\dot{\varphi}(0)=\omega_0,\theta(0)=\theta_0,\dot{\theta}(0)=\dot{\theta}_0,\rho(0)=\rho_0,\dot{\rho}(0)=\dot{\rho}_0.$$

The initial conditions (3.14) will be satisfied, if the functions $\psi_0(t),\psi_1(t),\eta_0(t),\dots$, satisfy the following conditions:

(3.15)
$$\psi_0(0)=\varphi_0\ ,\ \psi_1(0)=0\ ,\ \dot{\psi}_0(0)=\omega_0\ ,\ \dot{\psi}_1(0)=0\ ,$$
$$\eta_0(0)=\theta_0\ ,\ \eta_1(0)=0\ ,\ \dot{\eta}_0(0)=\dot{\theta}_0\ ,\ \dot{\eta}_1(0)=0\ ,$$
$$\xi_0(0)=\rho_0-l_{st}\ ,\ \xi_1(0)=0\ ,\ \dot{\xi}_0(0)=\dot{\rho}_0\ ,\ \dot{\xi}_1(0)=0.$$

Solving the first equation in (3.11) we get:

(3.16) $$\psi_0 = C_1 + C_2\, e^{-\frac{B_0}{J_0}t} + \frac{C_0}{B_0} t\,,$$

where the integration constants C_1 , C_2 were determined as: $C_1 = \varphi_0 - \frac{J_0}{B_0}\left(\frac{C_0}{B_0} - \omega_0\right)$, $C_2 = \frac{J_0}{B_0}\left(\frac{C_0}{B_0} - \omega_0\right)$. In a stationary regime of motion $T_M = T_W \Rightarrow a - b\omega_{st} = W R$, from which we get:

(3.17) $$\omega_{st} = \frac{a - W R}{b} = \frac{C_0}{B_0}$$

Assuming that $\varphi_0 = 0$, and using the result in (3.17) or (3.3), we get the following result for the function $\psi_0(t)$ in (3.16) and its derivatives:

(3.18) $$\psi_0(t) = -\frac{\Delta\omega}{\lambda}\left(1 - e^{-\lambda t}\right) + \omega_{st} t\,,$$

(3.19) $$\dot{\psi}_0 = \omega_{st} - \Delta\omega . e^{-\lambda t},\ \ddot{\psi}_0 = \lambda . \Delta\omega . e^{-\lambda t}\,,$$

where we let $\lambda = \frac{b}{M R^2}$, $\Delta\omega = \omega_{st} - \omega_0$.

We get the following result for the first equations in (3.12) and (3.13):

(3.20) $$\eta_0 = A_2 \cos(p_2 t)\ ,\ \xi_0 = A_3 \cos(p_3 t)\,,$$

where we let $A_2 = \theta_0$ and $A_3 = \rho_0 - l_{st}$ for the zero initial conditions $\dot{\theta}_0 = 0$, $\dot{\rho}_0 = 0$.

We can transform the second equations from (3.12) and (3.13), using the results from (3.19) and (3.20), to the following form:

(3.21) $$\ddot{\eta}_1 + p_2^{\,2}\eta_1 = h_1 A_2 \cos(p_2 t) + a_{21}\lambda . \Delta\omega . e^{-\lambda t} - \\ - a_{22}\, p_2^{\,2} A_2 A_3 \cos(p_2 t)\cos(p_3 t) + a_{23}\, p_2 p_3 A_2 A_3 \sin(p_2 t)\sin(p_3 t),$$

(3.22) $$\ddot{\xi}_1 + p_3^{\,2}\xi_1 = A_3\left(r_1 + \frac{1}{2}a_{33}\, p_2^{\,2} A_2^{\,2}\right)\cos(p_3 t) - \frac{1}{2}a_{33}\, p_2^{\,2} A_2^{\,2} A_3 \cos 2(p_2 t)\cos(p_3 t) + \\ + \frac{1}{2}a_{31}\, p_2^{\,2} A_2^{\,2} - \frac{1}{2}a_{31}\, p_2^{\,2} A_2^{\,2}\cos 2(p_2 t) + a_{32} e^{-\lambda t}\lambda . \Delta\omega . A_2 \cos(p_2 t).$$

In order to avoid a secular term in the solutions of (3.21) and (3.22), it is necessary that $h_1 A_2 = 0$ and $A_3\left(r_1 + \frac{1}{2}a_{33}\, p_2^{\,2} A_2^{\,2}\right) = 0$. With broadest initial conditions, these conditions will be satisfied if $h_1 = 0$ and $r_1 = -\frac{1}{2}a_{33}\, p_2^{\,2} A_2^{\,2} = -\frac{1}{2}m_2\, p_2^{\,2}\theta_0^{\,2}$. It follows ensues from this that:

(3.23) $$p_2^{\,2} = k_2^{\,2} = \frac{g}{l_{st}}\ \ ,\ \ p_3^{\,2} = k_3^{\,2} - \frac{1}{2}\mu m_2\, p_2^{\,2}\theta_0^{\,2} = \frac{c}{m_2} - \frac{1}{2}\frac{g}{l_{st}}\theta_0^{\,2}\,.$$

The solution of (3.22), for zero initial conditions (3.15), by means of a Duhamel integral is:

$$\xi_1 = \frac{1}{p_3}\int_0^t \sin p_3\left(t-\tau\right)\left[\frac{1}{2}a_{31}p_2^2A_2^2 - \frac{1}{2}a_{31}k_2^2A_2^2\cos 2p_2\tau + \right.$$

(3.24)
$$+a_{32}e^{-\lambda\tau}\lambda.\Delta\omega.A_2\cos\left(p_2\tau\right) - \frac{1}{4}a_{33}p_2{}^2A_2^2A_3\cos\left(2p_2-p_3\right)\tau -$$

$$\left. -\frac{1}{4}a_{33}p_2{}^2A_2^2A_3\cos\left(2p_2+p_3\right)\tau\right]d\tau.$$

Solving (3.24), substituting the result in (3.8) and using (3.20), we get the following result for the function $\rho(t)$, which describes the absolute stretch of the string:

$$\rho(t) = l_{st}\left(1+\frac{\theta_0^2\nu^2}{2}\right) - \frac{R\theta_0\lambda^2.\Delta\omega}{2p_3}\left[\frac{1}{\lambda^2+\left(p_3-p_2\right)^2} + \frac{1}{\lambda^2+\left(p_3+p_2\right)^2}\right]\sin\left(p_3t\right) +$$

$$\left[\left(\rho_0-l_{st}\right)\left(1+\frac{\theta_0{}^2}{8}\frac{\nu^2}{1-\nu^2}\right) + 2\theta_0^2l_{st}\frac{\nu^4}{1-4\nu^2} + \frac{R\theta_0\lambda.\Delta\omega}{2}\left(\frac{1-\nu}{\lambda^2+\left(p_3-p_2\right)^2} + \right.\right.$$

$$\left.\left.\frac{1+\nu}{\lambda^2+\left(p_3+p_2\right)^2}\right)\right]\cos\left(p_3t\right) - \frac{l_{st}\theta_0^2}{2}.\frac{\nu^2}{1-4\nu^2}\cos\left(2p_2t\right) + \frac{\theta_0^2\left(\rho_0-l_{st}\right)}{16}\frac{\nu}{1+\nu}\cos\left(2p_2+p_3\right)t -$$

(3.25)
$$-\frac{\theta_0^2\left(\rho_0-l_{st}\right)}{16}\frac{\nu}{1-\nu}\cos\left(2p_2-p_3\right)t - e^{-\lambda t}\frac{R\theta_0\lambda.\Delta\omega}{2}\left\{\left[\frac{1-\nu}{\lambda^2+\left(p_3-p_2\right)^2} + \right.\right.$$

$$\left.\left. + \frac{1+\nu}{\lambda^2+\left(p_3+p_2\right)^2}\right]\cos\left(p_2t\right) + \frac{\lambda}{p_3}\left[\frac{1}{\lambda^2+\left(p_3-p_2\right)^2} - \frac{1}{\lambda^2+\left(p_3+p_2\right)^2}\right]\sin\left(p_2t\right)\right\},$$

where we let $\nu = \dfrac{p_2}{p_3}$. It should be note that $1-4\nu^2 \neq 0$, i.e. $p_2 \neq 0{,}5.p_3 \Rightarrow m_2 \neq \dfrac{2cl_0}{g\left(6+\theta_0{}^2\right)}$

and $1-\nu \neq 0$, i.e. $p_2 \neq p_3 \Rightarrow m_2 \neq \dfrac{2cl_0}{g\theta_0{}^2}$.

Given the dynamic model in (Fig.1), the solution in (3.25) shows that the motion of the load along the axis $C\rho$ can be viewed as a superposition of free vibrations, caused by the initial conditions and parameters of the system, undamped forced vibrations and damped forced vibrations, dependent on the initial conditions. The motion takes place about a constant length of the string (the first two addends), which depends on the parameters of the system and on the initial conditions of the vibrating system.

We can present the solution of (3.21), using (3.20) and zero initial conditions (3.15), as the solution to the Duhamel integral:

(3.26)
$$\xi_1 = \frac{1}{p_2}\int_0^t \sin p_2\left(t-\tau\right)\left[a_{21}\lambda.\Delta\omega.e^{-\lambda t} - a_{22}p_2{}^2A_2A_3\cos\left(p_2\tau\right)\cos\left(p_3\tau\right) + \right.$$

$$\left. +a_{23}p_2\;p_3A_2A_3\sin\left(p_2\tau\right)\sin\left(p_3\tau\right)\right]d\tau.$$

We substitute the solution of (3.26), for $2p_2 \neq p_3 \Rightarrow m_2 \neq \dfrac{2cl_0}{g\left(6+\theta_0{}^2\right)}$, into (3.15) and using (3.20), we get the following result for the function $\theta(t)$:

$$\theta(t)=\left[\frac{R\lambda.\Delta\omega}{l_{st}}.\frac{1}{\lambda^2+p_2^2}+\theta_0\left(1-\frac{\rho_0-l_{st}}{l_{st}}.\frac{\nu(2-\nu)}{1-4\nu^2}\right)\right]\cos(p_2 t)-$$

$$\text{(3.27)}\qquad -\frac{R\lambda^2.\Delta\omega}{l_{st}p_2}.\frac{1}{\lambda^2+p_2^2}\sin(p_2 t)-\frac{\nu\theta_0(\rho_0-l_{st})}{2l_{st}}\left[\frac{1}{1-2\nu}\cos(p_2-p_3)t+\right.$$

$$\left.+\frac{1}{1+2\nu}\cos(p_2+p_3)t\right]-e^{-\lambda t}\frac{R\lambda.\Delta\omega}{l_{st}}.\frac{1}{\lambda^2+p_2^2}$$

The result in (3.27) shows that the deviation of the elastic pendulum is a result of the superposition of vibrations with natural frequency p_2, vibrations with frequencies p_2-p_3 and p_2+p_3, caused by the initial conditions of the generalized coordinates of the elastic mathematical pendulum (θ_0,ρ_0), and of the suspension point ($\Delta\omega$). The last addend in (3.27) has a negligible effect because $e^{-\lambda t}\to 0$ fast.

We can transform the second equation in (3.11), using (3.20), as follows:

$$\text{(3.28)}\qquad \ddot{\psi}_1+\frac{B_0}{J_0}\dot{\psi}_1=b_{11}\cos(p_2 t)+b_{12}\cos(3p_2 t)+b_{13}\cos(p_2-p_3)t+$$

$$+b_{14}\cos(p_2+p_3)t+b_{15}\cos(3p_2-p_3)t+b_{16}\cos(3p_2+p_3)t\ ,$$

where we let: $b_{11}=\frac{A_2p_2^2}{J_0}\left(\frac{1}{4}a_{15}A_2^2-a_{14}\right)$, $\quad b_{12}=-\frac{A_2^2p_2^2a_{15}}{4J_0}$,

$$b_{13}=\frac{A_2A_3}{2J_0}\left[a_{13}p_2p_3-a_{11}p_3^2-p_2^2\left(a_{12}-\frac{7}{4}a_{16}A_2^2\right)\right],\ b_{15}=-\frac{a_{16}A_2^3A_3p_2^2}{8J_0}$$

$$b_{14}=-\frac{A_2A_3}{2J_0}\left[a_{13}p_2p_3+a_{11}p_3^2+p_2^2\left(a_{12}-\frac{7}{4}a_{16}A_2^2\right)\right],\ b_{16}=-\frac{a_{16}A_2^3A_3p_2^2}{8J_0}\ .$$

The solution of the canonical differential equation (3.28) with zero initial conditions (3.15) by means of the Duhamel integral is:

$$\psi_1=\frac{1}{\lambda}\int_0^t\left(1-e^{-\lambda(t-\lambda)}\right)q(\tau)d\tau=\sum_{i=1}^{6}I_i=$$

$$\text{(3.29)}\qquad =\sum_{i=1}^{6}\frac{1}{\lambda}\int_0^t b_{1i}\left(1-e^{-\lambda(t-\tau)}\right)\cos(\alpha_i\tau)d\tau=K_{i1}\cos(\alpha_i t)+K_{i2}\sin(\alpha_i t)+e^{-\lambda t}K_{i3}\ ,$$

where we let $\alpha_1=p_2$, $\alpha_2=3p_2$, $\alpha_3=p_2-p_3$, $\alpha_4=p_2+p_3$, $\alpha_5=3p_2-p_3$, $\alpha_6=3p_2+p_3$, $K_{i1}=-\frac{b_{1i}}{\lambda^2+\alpha_i^2}$, $K_{i2}=-\frac{\lambda}{\alpha_i}K_{i1}, K_{i3}=-K_{i1}$.

We get the following result for the angular velocity ω of the disk, using (3.29), (3.19), and (3.8):

$$\omega=\omega_{st}+e^{-\lambda t}A_0+A_{11}\cos(p_2t)+A_{12}\sin(p_2t)+A_{21}\cos(3p_2t)+$$

$$+A_{22}\sin(3p_2t)+A_{31}\cos((p_2-p_3)t)+A_{32}\sin((p_2-p_3)t)+$$

$$\text{(3.30)}\qquad +A_{41}\cos((p_2+p_3)t)+A_{42}\sin((p_2+p_3)t)+A_{51}\cos((3p_2-p_3)t)+$$

$$+A_{52}\sin((3p_2-p_3)t)+A_{61}\cos((3p_2+p_3)t)+A_{62}\sin((3p_2+p_3)t),$$

where $A_0=-\Delta\omega-\mu\lambda\sum_{i=1}^{6}K_{i3}=-\Delta\omega-\frac{m_2\lambda\theta_0}{MR}\left\{l_{st}p_2^{\,2}\left[\frac{1+0,25\theta_0^{\,2}}{\lambda^2+p_2^{\,2}}-\frac{0,25\theta_0^{\,2}}{\lambda^2+(3p_2)^2}\right]+\right.$

$$+0,5(\rho_0-l_{st})\left[\frac{(p_2-p_3)^2+1,75p_2^{\,2}\theta_0^{\,2}}{\lambda^2+(p_2-p_3)^2}+\frac{(p_2+p_3)^2+1,75p_2^{\,2}\theta_0^{\,2}}{\lambda^2+(p_2+p_3)^2}\right]-$$

$$\left.-0,125p_2^{\,2}\theta_0^{\,2}(\rho_0-l_{st})\left[\frac{1}{\lambda^2+(3p_2-p_3)^2}+\frac{1}{\lambda^2+(3p_2+p_3)^2}\right]\right\},$$

$$A_{11}=\mu p_2K_{12}=\lambda\frac{p_2^{\,2}}{\lambda^2+p_2^{\,2}}\frac{m_2l_{st}\theta_0}{MR}\left(1+0.25\theta_0^{\,2}\right),$$

$$A_{12}=-\mu p_2K_{11}=p_2\frac{p_2^{\,2}}{\lambda^2+p_2^{\,2}}\frac{m_2l_{st}\theta_0}{MR}\left(1+0.25\theta_0^{\,2}\right),$$

$$A_{21}=\mu(3p_2)K_{22}=-\frac{\lambda}{36}\frac{(3p_2)^2}{\lambda^2+(3p_2)^2}\frac{m_2l_{st}\theta_0^{\,2}}{MR},$$

$$A_{22}=-\mu(3p_2)K_{21}=-\frac{3p_2}{36}\frac{(3p_2)^2}{\lambda^2+(3p_2)^2}\frac{m_2l_{st}\theta_0^{\,2}}{MR}$$

$$A_{31}=\mu(p_2-p_3)K_{32}=\frac{\lambda}{2}\frac{1}{\lambda^2+(p_2-p_3)^2}\frac{m_2(\rho_0-l_{st})\theta_0}{MR}\left[(p_2-p_3)^2+1.75p_2^{\,2}\theta_0^{\,2}\right],$$

$$A_{32}=-\mu(p_2-p_3)K_{31}=\frac{(p_2-p_3)}{2}\frac{1}{\lambda^2+(p_2-p_3)^2}\frac{m_2(\rho_0-l_{st})\theta_0}{MR}\left[(p_2-p_3)^2+1.75p_2^{\,2}\theta_0^{\,2}\right],$$

$$A_{41}=\mu(p_2+p_3)K_{42}=\frac{\lambda}{2}\frac{1}{\lambda^2+(p_2+p_3)^2}\frac{m_2(\rho_0-l_{st})\theta_0}{MR}\left[(p_2+p_3)^2+1.75p_2^{\,2}\theta_0^{\,2}\right],$$

$$A_{42}=-\mu(p_2+p_3)K_{41}=\frac{(p_2+p_3)}{2}\frac{1}{\lambda^2+(p_2+p_3)^2}\frac{m_2(\rho_0-l_{st})\theta_0}{MR}\left[(p_2+p_3)^2+1.75p_2^{\,2}\theta_0^{\,2}\right],$$

$$A_{51}=\mu(3p_2-p_3)K_{52}=-\frac{\lambda}{8}\frac{p_2^{\,2}}{\lambda^2+(3p_2-p_3)^2}\frac{m_2\theta_0^{\,3}(\rho_0-l_{st})}{MR},$$

$$A_{52}=-\mu(3p_2-p_3)K_{51}=-\frac{(3p_2-p_3)}{8}\frac{p_2^{\,2}}{\lambda^2+(3p_2-p_3)^2}\frac{m_2\theta_0^{\,3}(\rho_0-l_{st})}{MR},$$

$$A_{61}=\mu(3p_2+p_3)K_{62}=-\frac{\lambda}{8}\frac{p_2^{\,2}}{\lambda^2+(3p_2+p_3)^2}\frac{m_2\theta_0^{\,3}(\rho_0-l_{st})}{MR},$$

$$A_{62}=-\mu(3p_2+p_3)K_{61}=-\frac{(3p_2+p_3)}{8}\frac{p_2^{\,2}}{\lambda^2+(3p_2+p_3)^2}\frac{m_2\theta_0^{\,3}(\rho_0-l_{st})}{MR}.$$

It follows from (3.30) that weak harmonic vibrations are superimposed on the uniform motion of the disk, which is determined by ω_{st} (3.17). The addend $e^{-\lambda t}A_0$ has a weak, short-term effect on the motion of the disc because $e^{-\lambda t}\to 0$ fast.

The laws governing the change of the generalized coordinates, which were derived in this paper, can be used to determine and analyze the dynamic load on the string: $F = c(\rho - l_0)$.

4. Numerical Example

The theoretical results derived in this work were used to analyze the motion of an overhead traveling crane with a maximum safe working load of 200 kN, moving at a constant speed of 25 m/s.
The following numerical values were used in the experiment:

$m_1 = 250{,}8\,\text{kg}$, $m_2 = 200\,\text{kg}, m_3 = 5949{,}2\,\text{kg}$,

$R = 0{,}16\,m$, $W = 1216\,\text{N}$, $l_0 = 10\,m$,

$c = 246000\,\text{N/m}$, $\omega_0 = 2{,}6\,\text{s}^{-1}$,

$\theta_0 = -0{,}1\,\text{rad}$, $\rho_0 = 10\,\text{m}$.

The results, obtained after substituting the aforementioned numerical values into the analytical solution of the differential equations, were compared with the numerical solution of the system of differential equations (2.6), which was found using a fourth order modified Hamming predictor-corrector method. The change in angular velocity ω is shown on fig.2, the deviation of the string θ on fig.3, the change in length of the string on fig.4.

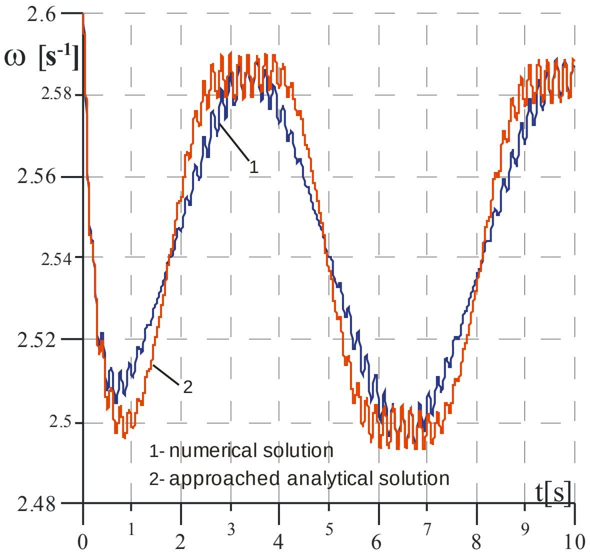

Fig.2 Angular velocity $\omega = \omega(t)$

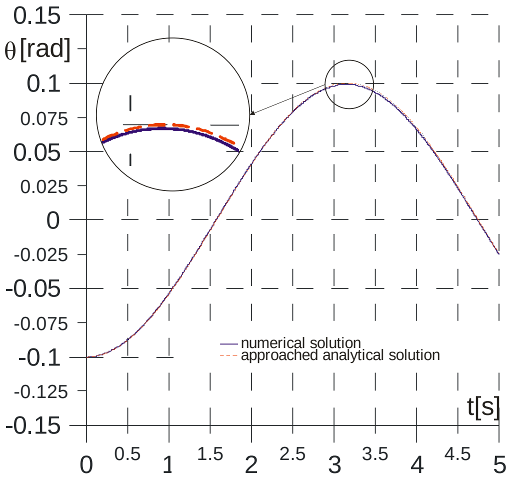

Fig.3 Deviation of the string $\theta = \theta(t)$

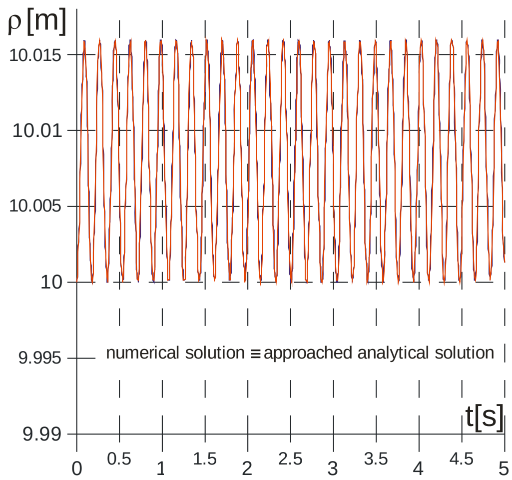

Fig.4 Length of the string $\rho = \rho(t)$

5. Results

A dynamic model of mechanical systems with suspended weights, including mechanisms used for moving and hoisting, was developed. The dynamic model is discrete with three degrees of freedom and includes a homogeneous disc, which rolls without slipping along a horizontal plane and whose center is the suspension point of an elastic mathematical pendulum. The motion of the system can be described using three non-linear differential equations, which were derived using Lagrange's method. The analytical solution to the system of differential equations was found using the method of the small parameter with higher accuracy then the solution in [16]. Numerical values were substituted into the analytical solution in order to analyze the motion of an overhead traveling crane, which is used in industry. The results of the experiment show a very good agreement between the analytical solution and the numerical solution, which was obtained using a fourth order modified Hamming predictor-corrector method. The analytical solution to the system of differential equations of motion can be used in different areas of the engineering practice because it is a necessary condition for performing qualitative dynamic analysis and for solving dynamic synthesis problems.

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ЗА ДИНАМИКАТА НА ЕЛАСТИЧНО МАТЕМАТИЧНО МАХАЛО С ПОДВИЖНА ТОЧКА НА ОКАЧВАНЕ

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***Ключови думи**: Еластично математично махало, аналитични методи.*

***Резюме**: Разглежда се еластично математично махало с подвижна точка на окачване. Товарът е представен като материална точка и е окачен на еластична безмасова нишка. Точката на окачване е реализирана в центъра на хомогенен диск, който се търкаля без плъзгане по хоризонтална равнина. Дискът е модел на механизъм за преместване в стационарен режим на движение. Посредством методите на нелинейната механика са определени закона на движение на еластичното математично махало и динамичното натоварване на гъвкавата връзка.*