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COMPLEX BEHAVIOR OF DOUBLE INVERTED PENDULUM WITH A VERTICALLY OSCILLATING SUSPENSION POINT

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Key words: *double inverted pendulum, bifurcation diagrams, complex behavior.*

Summary: *We study in this paper the complex behavior of double inverted pendulum with a vertically oscillating suspension point. The system of interest is derived as a model of concrete mechanical phenomena. The equations representing this model are nonlinear ordinary differential ones and are treated as nonlinear dynamical system. The results show that stability of inverted state depends from amplitudes of oscillation of suspended point. Bifurcation diagrams are found which locate regions of periodic and quasi-periodic motion within the parameter space of the system.*

1. Introduction

Dynamical regimes observed in nonlinear systems are classified by their complexity. The most simple nontrivial regime is a periodic one which is typical for conservative systems with one degree freedom. In autonomous dissipative dynamical systems periodic solutions appear when there are mechanisms of energy supply and energy dissipation present in the system. Thus, a periodic motion with a certain amplitude appears as a structurally stable regime. Often this motion is called self-sustained oscillations.

The next complex dynamical regime is quasi-periodicity. This behavior appears in elementary linear conservative systems with two degrees of freedom. There is another way to construct a quasi-periodic state in a dissipative system. One starts with a simple stable steady state which state can lose stability when the parameters of the system are changed. Hence, a stable limit cycle via an Andronov-Hopf bifurcation occurs. With a further change of parameter(s) the periodic motion can become unstable and secondary Andronov-Hopf (also called Neimark-Sacker) bifurcation take place. As a result, a quasi-periodic motion with two frequences ω_1 and ω_2 can appear. Assuming that further secondary bifurcations can occur, one can image the appearance of quasi-periodic motions of higher order- *chaotic regime* [13].

For complex dynamics the calculation of the bifurcation diagram is of major significance, as it provides a practical, numerically feasible tool for distinguishing periodic, quasi-periodic and chaotic motions.

The pendulum is interesting as a paradigm of contemporary nonlinear dynamics and, more importantly, because the differential equation of the pendulum is frequently encountered in various branches of modern mechanics.

The dynamic stabilization of double inverted pendulum with a vertically oscillating point (see Fig. 1) possesses an interesting behavior. It is well know that, the double inverted pendulum becomes stabilized after its instability, destabilizes again, and so forth ad infinitum. This type of dynamic stability probably was first explained and investigated experimentally in detail by Pjotr Kapitza in 1951 [2]. Now, this mechanical device in Russia is widely known as "Kapitza's pendulum" [3].

After Kapitza, several investigators have sought to develop nonlinear dynamical models of the pendulum and inverted pendulum when the suspension points oscillating one [4-9]. The double inverted pendulums with a vertically suspension point can be used to model many mechanical [4] and biological (biped locomotion) systems [10]. In the recent years, the nonlinear chaotic dynamics of the inverted pendulums has attracted considerable attention [1], [11-12]. In [1] a parametrically forced pendulum with the vertically oscillating suspension point is considered. It is found that the inverted state stabilizes via alternating "reverse" subcritical pitch-fork and period-doubling bifurcations, while it destabilizes via "normal" supercritical period-doubling and pitch-fork bifurcations.

In this paper we are interested in the stability of the inverted state when the suspension point makes vertically oscillations. Here the Lagrange's formula can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=-\frac{\partial \Pi}{\partial q_j}, \tag{1}$$

where $j=(1, 2)$, T is the kinetic energy of the system, Π is the potential energy of the system and $q_j, \dot{q}_j$ are the generalized coordinates and velocities, respectively. For double inverted pendulum with vertically oscillating suspension point, T and Π have the form

$$T=\frac{m_1}{2}\left(l_1^2\dot{\varphi}_1^2+2l_1\dot{\varphi}_1\dot{r}\sin\varphi_1+\dot{r}^2\right)+\frac{m_2}{2}\Big[l_1^2\dot{\varphi}_1^2+l_2^2\dot{\varphi}_2^2+2l_1l_2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1-\varphi_2)+ \\ +2\dot{r}\left(l_1\dot{\varphi}_1\sin\varphi_1+l_2\dot{\varphi}_2\sin\varphi_2\right)+\dot{r}^2\Big], \tag{2}$$

$$\Pi=m_1g(r-l_1\cos\varphi_1)+m_2g(r-l_1\cos\varphi_1-l_2\cos\varphi_2) \tag{3}$$

After substituting of (2) and (3) into (1) and accomplishing some transformations, we obtain

$$\begin{gathered} l_1\ddot{\varphi}_1+\ddot{r}\sin\varphi_1+\frac{m_2}{m_1}\left[l_1\ddot{\varphi}_1+l_2\ddot{\varphi}_2\cos(\varphi_1-\varphi_2)+l_2\dot{\varphi}_2^2\sin(\varphi_1-\varphi_2)+\ddot{r}\sin\varphi_1\right]= \\ =-g\sin\varphi_1\left(1+\frac{m_2}{m_1}\right), \\ l_2\ddot{\varphi}_2+l_1\ddot{\varphi}_1\cos(\varphi_1-\varphi_2)-l_1\dot{\varphi}_1^2\sin(\varphi_1-\varphi_2)+\ddot{r}\sin\varphi_2=-g\sin\varphi_2, \end{gathered} \tag{4}$$

where $r(t)=y_A$ is the vertically deviation of the suspension point, $\varphi_{1,2}$ are the angular displacements measured from the downward vertical (see Fig.1), $m_{1,2}$ are the masses attached to two ends of the light rods (its masses can be negligible) of lengths $l_{1,2}$, respectively.
If the angles $\varphi_{1,2}$ are small, (4) can be linearized by let

$$\cos(\varphi_1-\varphi_2)=1,\ \cos\varphi_1=1,\ \cos\varphi_2=1,\ \sin(\varphi_1-\varphi_2)=\varphi_1-\varphi_2,\ \sin\varphi_1=\varphi_1,\ \sin\varphi_2=\varphi_2. \tag{5}$$

After substitution of (5) into (4) and accomplishing some transformations, the system (4) takes the form

$$\begin{aligned}\ddot{\varphi}_1 &= -\left(A_1+A_2\ddot{r}\right)\varphi_1+\left(A_3+A_4\ddot{r}\right)\varphi_2-A_5\dot{\varphi}_1^2(\varphi_1-\varphi_2)-A_6\dot{\varphi}_2^2(\varphi_1-\varphi_2),\\ \ddot{\varphi}_2 &= \left(A_7+A_8\ddot{r}\right)\varphi_1-\left(A_9+A_{10}\ddot{r}\right)\varphi_2+A_{11}\dot{\varphi}_1^2(\varphi_1-\varphi_2)+A_{12}\dot{\varphi}_2^2(\varphi_1-\varphi_2),\end{aligned} \tag{6}$$

where

$$\begin{aligned}&A_1=\frac{(m_1+m_2)g}{l_1(m_1+2m_2)},\ A_2=\frac{m_1+m_2}{l_1(m_1+2m_2)},\ A_3=\frac{m_2g}{l_1(m_1+2m_2)},\ A_4=\frac{m_2}{l_1(m_1+2m_2)},\\ &A_5=\frac{m_2}{m_1+2m_2},\ A_6=\frac{m_2l_2}{l_1(m_1+2m_2)},\ A_7=\frac{(m_1+m_2)g}{l_2(m_1+2m_2)},\ A_8=\frac{m_1+m_2}{l_2(m_1+2m_2)},\\ &A_9=\frac{(m_1+3m_2)g}{l_2(m_1+2m_2)},\quad A_{10}=\frac{m_1+3m_2}{l_2(m_1+2m_2)},\quad A_{11}=\frac{l_1(m_1+3m_2)}{l_2(m_1+2m_2)},\quad A_{12}=\frac{m_2}{m_1+2m_2}.\end{aligned} \tag{7}$$

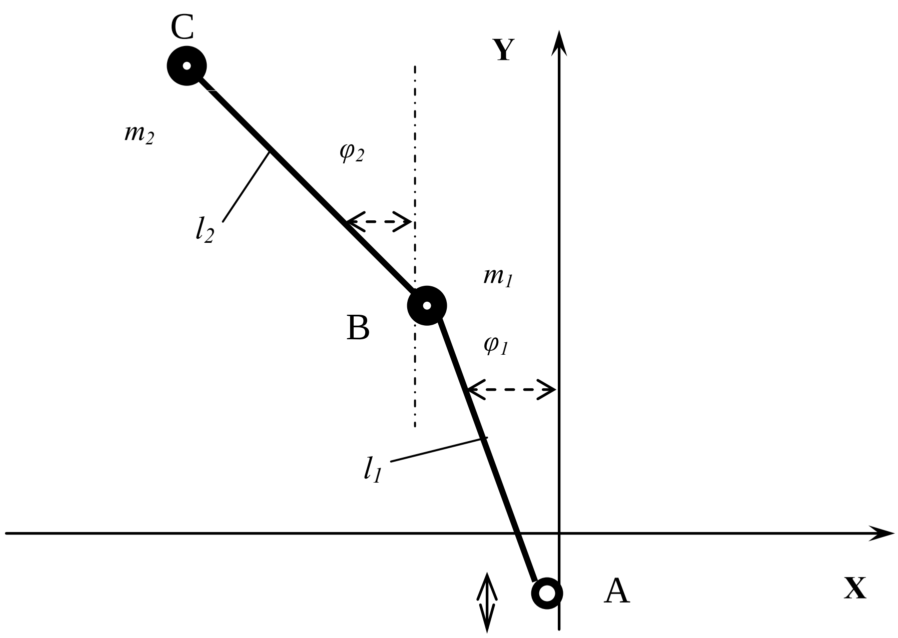

Figure 1. Illustration of the double inverted pendulum with vertically oscillating suspension point.

In this paper, we undertake an investigation of the dynamics of the double inverted pendulum with the vertically oscillating suspension point. In particular, we consider the special case when $r(t)$ has the form.

(8) $$r(t) = a_1 \sin \omega_1 t + a_2 \sin \omega_2 t\,.$$

where ω_1 and ω_2 are frequencies. It is well-known that (8) is periodic only if the ratio of the frequencies is rational, i.e. when

(9) $$\frac{\omega_1}{\omega_2} = \frac{\alpha_2}{\alpha_1},$$

with integers α_1 and α_2. In this case the period is $T = \alpha_2 \frac{2\pi}{\omega_1} = \alpha_1 \frac{2\pi}{\omega_2}$. By contrast, if the ratio of frequencies is an irrational number, (9) cannot be valid for any pair of integers α_1 and α_2, and the vertical motion of sustained point A is quasi-periodic.

If we differentiate twice the equation (8) and substitute into (6) we obtain

(10)
$$\begin{aligned}
\ddot{\varphi}_1 &= -\left[A_1 - A_2\left(a_1\omega_1^2 \sin \omega_1 t + a_2\omega_2^2 \sin \omega_2 t\right)\right]\varphi_1 + \left[A_3 - A_4\left(a_1\omega_1^2 \sin \omega_1 t + a_2\omega_2^2 \sin \omega_2 t\right)\right]\varphi_2 - \\
&\quad - A_5 \dot{\varphi}_1^2 (\varphi_1 - \varphi_2) - A_6 \dot{\varphi}_2^2 (\varphi_1 - \varphi_2), \\
\ddot{\varphi}_2 &= \left[A_7 - A_8\left(a_1\omega_1^2 \sin \omega_1 t + a_2\omega_2^2 \sin \omega_2 t\right)\right]\varphi_1 - \left[A_9 - A_{10}\left(a_1\omega_1^2 \sin \omega_1 t + a_2\omega_2^2 \sin \omega_2 t\right)\right]\varphi_2 + \\
&\quad + A_{11} \dot{\varphi}_1^2 (\varphi_1 - \varphi_2) + A_{12} \dot{\varphi}_2^2 (\varphi_1 - \varphi_2),
\end{aligned}$$

In our previous paper [14], we obtain analytical results for stabilization of the inverted state as a function of frequencies ω_1 and ω_2, and lengths $l_{1,2}$. Also, we presented the numerical results for stabilized inverted state and maximal Lyapunov exponent. In the present paper, we continue the investigation of stability of inverted state as function of magnitudes of oscillation of suspended point, a_1 and a_2, by numerical simulations- bifurcation diagrams.

2. Numerical analysis

In previous section we obtained the model (eq. (10)) that we will use in our numerical analysis. Involving in (9) dimension parameters present the characteristic values of the inverted pendulum moving system. These are some corresponding values of the masses attached to two ends of the light rods (they masses can be negligible) $m_{1,2}$, length rods $l_{1,2}$, gravity acceleration g, frequencies $\omega_{1,2}$ and dimensionless amplitudes $a_{1,2}$. In this work, the characteristic values are

(11)
$$m_1 = 0.35kg,\, m_2 = 0.55kg,\, l_1 = 0.25m,\, l_2 = 0.35m,\, a_1 = 0.01; 0.03,\, a_2 \in [0.01, 0.04]$$

and

(12) $$\alpha = \frac{\omega_1}{\omega_2} = \frac{13}{19}.$$

In Figures 2a and 2b, we show the bifurcation diagrams of system (10): values of φ_1 and φ_2 coordinates are plotted against a_2 regarded as a continuously varying bifurcation (control) parameter for $a_1 = 0.01$. The white zones (when $a_2 \in [0.027, 0.029]$), seen in Fig. 2c, correspond to inverse period-doubling bifurcations. It is evident that inverted state of the system (10) when the suspension point makes vertically oscillations (see Eq. (8)) is stable. Here we note that for all simulations (see Figs.2-4) the initial conditions were $\varphi_1(0) = \dot{\varphi}_1(0) = \dot{\varphi}_2(0) = 0, \varphi_2 = 0.02$.

On the next Figure 3, the bifurcation diagrams of the angle φ_1 and the angle φ_2 are shown at α rational number also, i.e. $\omega_1 = 13, \omega_2 = 19$. It is easy to see that now the inverted state is also stable. Comparing Figure 3a and Figure 3b, we conclude that inverse bifurcations are characteristic of φ_1 behavior. It is interesting to note here that as one increases a_2 from $a_2 = 0.01$ (till $a_2 = 0.029$) amplitudes of oscillations rapidly increase and decrease.

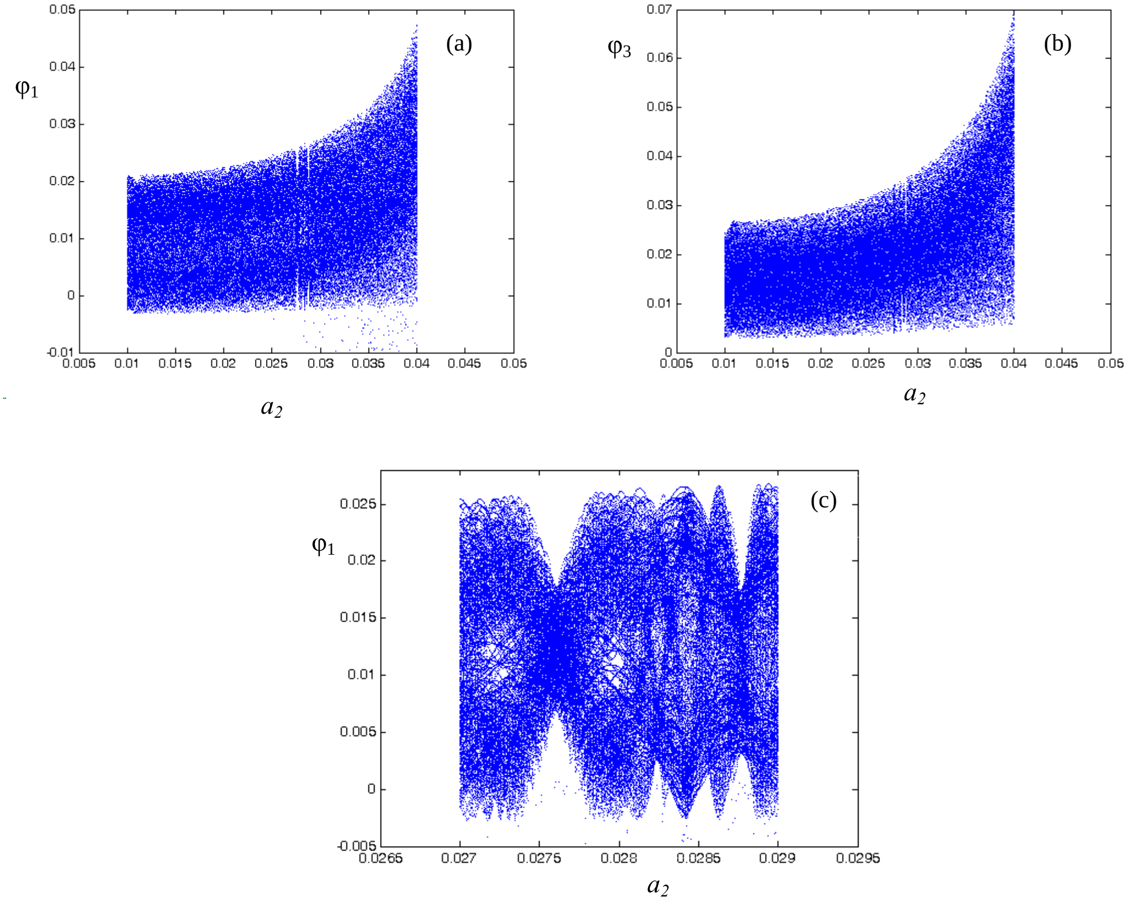

Figure 2. (a), (b) Bifurcation diagrams of system (10) at $a_1 = 0.01, a_2 \in [0.01, 0.04]$ and (c) detailed part of the diagram for φ_1 when $a_2 \in [0.027, 0.029]$.

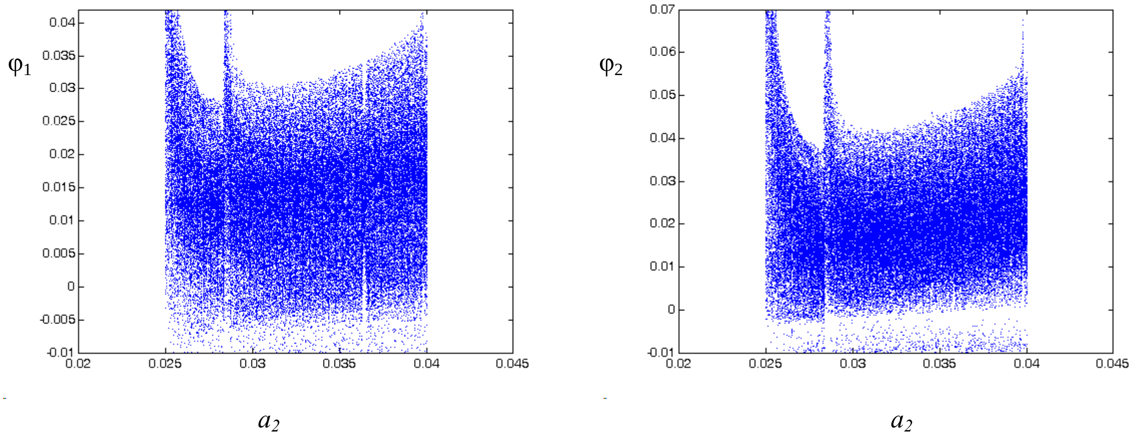

Figure 3. Bifurcation diagrams of system (10) at $a_1 = 0.03$, $a_2 \in [0.01, 0.04]$.

Figure 4a shows a Poincare section (stroboscopic portrait) of the state space taken at the phase when driving force has its largest value, that is, when $\sin\omega_1 t = 1$ and $\sin\omega_2 t = 1$. What makes the oscillator interesting is that oscillations depends on the amplitudes a_1 and a_2. A natural trajectory is illustrated in Figure 4b. Here we note that quasi-oscillations (two dimensional tore) take place.

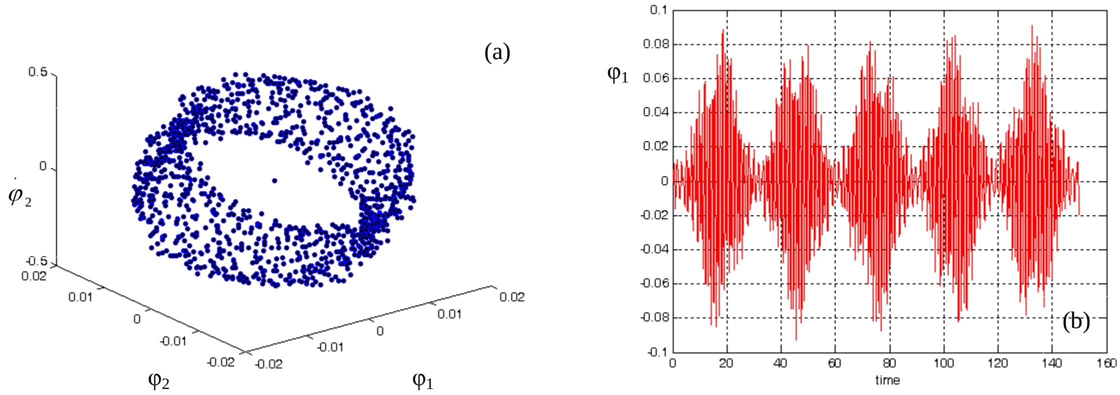

Figure 4. (a) The Poincare section of the state space when $a_1 = 0.03$ and $a_2 = 0.025$ and (b) natural solution for φ_1.

4. Summary and conclusions

In this paper we investigate the problem of complex behavior of double inverted pendulum with a vertically oscillating suspension point. The system of interest is derived as a model of concrete mechanical phenomena. The equations representing this model are nonlinear ordinary differential ones and are treated as nonlinear dynamical system. The numerical calculations show that dynamical bifurcation behavior of the system depends from parameters a_1 and a_2. Here we note that in all simulations the parameters and the initial conditions were $m_1 = 0.35kg, m_2 = 0.55kg, l_1 = 0.25m, l_2 = 0.35m,\quad a_1 = 0.01;0.03, a_2 \in [0.01, 0.04]$,

$\omega_1 = 13, \omega_2 = 19, \varphi_1(0) = \dot{\varphi}_1(0) = \dot{\varphi}_2(0) = \ddot{r}(0) = 0, \varphi_2 = 0.02, \dot{r}(0) = 1$. The basic conclusions have mostly dynamical character, and they are:

1. Regarding the numerical calculations outlined above (see Figs. 2-4), the stability of the double pendulum inverse state substantially depends on the values of $a_{1,2}$.
2. When $a_1 = 0.01; 0.03, a_2 \in [0.01, 0.04]$, the system (10) has periodic and quasi-periodic solutions, and strange non-chaotic attractors take place

References

[1] Kim, S., Kim, Y., Dynamic stabilization in the double-well Duffing oscillator, Physical Review E, vol. 61, No 6, pp. 6517-6520, 2000.

[2] Kapitza, P., In: Collected papers of P.L. Kapitza (Ed. D. Ter Haar), Pergamon, London, 1965.

[3] Butikov, E., On the dynamic stabilization of an inverted pendulum, Am. J. Phys., vol. 69, No 7, pp. 755-768, 2001.

[4] Strijak, T., Investigation methods of the dynamical systems in the pendulum form, Nauka, Alma-Ata, 1981 (in Russian).

[5] Брадистилов, Г., Положение системы трех последовательно соединенных математических маятников, находящихся в одной плоскости, при ее периодическом движении вокруг положения устойчивого равновесия, Прикл. Математика и механика, т. 19, стр. 113-118, 1955. (Bradistilov, G., Polozhenie sistemы treh posledovatelyno soedinennыh matematicheskih mayatnikov, nahodyashtihsya v odnoy ploskosti, pri ee periodicheskom dvizhenii vokrug polozhenia ustoychivogo ravnovesia, Prikl. Matematika i mehanika, t. 19, str. 113-118, 1955.)

[6] Брадистилов, Г., Положение тройного матемагического маятника в плоскости при его периодическом и асимптотическом движении вокруг положения равновесия, Прикл. Математика и механика, т. 19, стр. 485-492, 1955. (Bradistilov, G., Polozhenie troynogo matemagicheskogo mayatnika v ploskosti pri ego periodicheskom i asimptoticheskom dvizhenii vokrug polozhenia ravnovesia, Prikl. Matematika i mehanika, t. 19, str. 485-492, 1955.)

[7] Neimark, U., Kogan, N., Saveliev, V., Dynamical models of control theory, Nauka, Moscow, 1985 (in Russian).

[8] Tondl, A., Kotek, V., Kratochvil, C., Analysis of an autoparametric system, Eng. Mechanics, vol. 6, No 1, pp. 37-44, 1999.

[9] Nikolov, S., Bachvarov, S., Dynamic behavior of inverted pendulum with a cycloidal oscillating suspension point, Eng. Mechanics, vol. 11, No 3, pp. 201-214, 2004.

[10] Furusho, J., Masubuchi, M., A theoretically motivated reduced order model for the control of dynamic biped locomotion, J. of Dynamic Systems, Measurement, and Control, vol. 109, pp. 155-161, 1987.

[11] Kim, S., Hu, B., Bifurcations and transitions to chaos in an inverted pendulum, Physical Review E, vol. 58, No 3, pp. 3028-3035, 1998.

[12] DeSerio, R., Chaotic pendulum: The complete attractor, Am. J. Phys, vol. 71, No 3, pp. 250-257, 2003.

[13] Fuedel, U., Kuznetsov, S., Pikovsky, A., Strange nonchaotic attractors: dynamics between order and chaos in quasi-periodically forced systems, World Scientific, 2006.

[14] Nikolov, S., Bachvarov, S., Zlatanov, V., Klochkov, L., Stabilization and chaotic behavior of double inverted pendulum with a vertically oscillating suspension point, Proceedings of the 30-th Jubilee Int. Conference of TU Sofia, June 2004, Sozopol, Bulgaria, pp.140-146, 2005.