Mechanics Transport Communications

Academic journal

issn 1312-3823
issue 1, 2010
article № 0472
http://www.mtc-aj.com

STABILITY OF NONLINEAR MECHANICAL SYSTEM WITH TWO DEGREES OF FREEDOM

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Key words: *stability, nonlinear system, two degrees of freedom*
Abstract: *In this paper we investigate how the inclusion of nonlinear restoring force alters the stability properties of the two mass mechanical system. The consequences of a nonlinear restoring force on the stability of this system are analysed using Routh-Hurwitz criterion for stability. Our analytical calculations predict that nonlinear fragment acts as a key parameter. This is confirmed by numerical simulations*

1. Introduction

Nonlinear dynamics studies some characteristics not observed in linear systems. Nonlinear behavior (movement or evolution) is typical for different kind mechanical, biological and chemical systems. Investigation of nonlinear mechanical systems is an important and very active area [1]. Dynamical systems from nonlinear ordinary differential equations are the most accurate mathematical way to describe a smooth continuous evolution. One of the basic questions in studying dynamical systems is the construction of invariants that allow us to classify qualitative types of dynamical evolution, to distinguish between qualitatively different dynamics, and to study transitions between different types. More specifically, it is important to find out when a certain dynamic behavior is stable under small perturbation, as well as to understand the various scenarios of instability [2, 3].

The concept of stability (in terms of Lyapunov's original definition [4], i.e. Lyapunov stability) is usually referred to the qualitative behavior of motions relative to an invariant set (resp. an equilibrium state). In his famous doctoral dissertation [4], Lyapunov developed two general methods for the stability analysis of an equilibrium: *Lyapunov's direct method* (or also called *The second method of Lyapunov*) and *the Indirect method of Lyapunov* (or also called *The first method of Lyapunov*) [5]. In the process of discovering the first method, involving the Lyapunov matrix equation, Lyapunov established some important stability results for linear systems. Note that equivalent of these results are independently discovered results by Routh (five years earlier) and Hurwitz (three years later).

In recent years, the main interest of the researchers have been focused on the response of a nonlinear system, which is near the onset of dynamical instabilities, to small periodic perturbations, small-signal amplification of bifurcating system [6, 7], periodic multistability [8], and other periodic driving induced behaviors in excitable or oscillatory systems [9, 10].

The results obtained in these papers motivate us to investigate here the stability behavior of a two mass mechanical system (depicted in Figure 1), which has the form

(1)
$$m_1 \ddot{x}_1 + c_1 x_1 + c(x_1 - x_2) + k_1 \dot{x}_1 + k\left(\dot{x}_1 - \dot{x}_2\right) = 0,$$
$$m_2 \ddot{x}_2 + c_2 x_2 + c(x_2 - x_1) + k_2 \dot{x}_2 + k\left(\dot{x}_2 - \dot{x}_1\right) = 0,$$

where x_i $(i = 1, 2)$ is the displacement for mass m_i; c, c_i are the spring constants; and k, k_i are the viscous damping coefficients. We assume that $m_i > 0, c > 0, c_i > 0, k \geq 0, k_i \geq 0,$ where $k + k_1 + k_2 > 0.$

Most oscillating mechanical systems are not exactly linear but are approximately linear when the oscillation amplitude is small. In the case of a body on a spring, the restoring force F_R might actually have the form

(2)
$$F_R = Cx + \Lambda x^3$$

which is approximated by the linear formula $F_R = cx$ - when the displacement x is small. The constant Λ is a measure of the strength of the nonlinear effect. It is well-known that if $\Lambda < 0$, then F_R is less than its linear approximation and the spring is said to be softening as x increases. Conversely, if $\Lambda > 0$, then the spring is hardening as x is increases. The formula (2) is typical of nonlinear restoring forces that are symmetrical about $x = 0$. If the restoring force is unsymmetrical about $x = 0$, the leading correction to the linear case will be a term in x^2, i.e.

(3)
$$F_R = Cx + \mathrm{B}x^2.$$

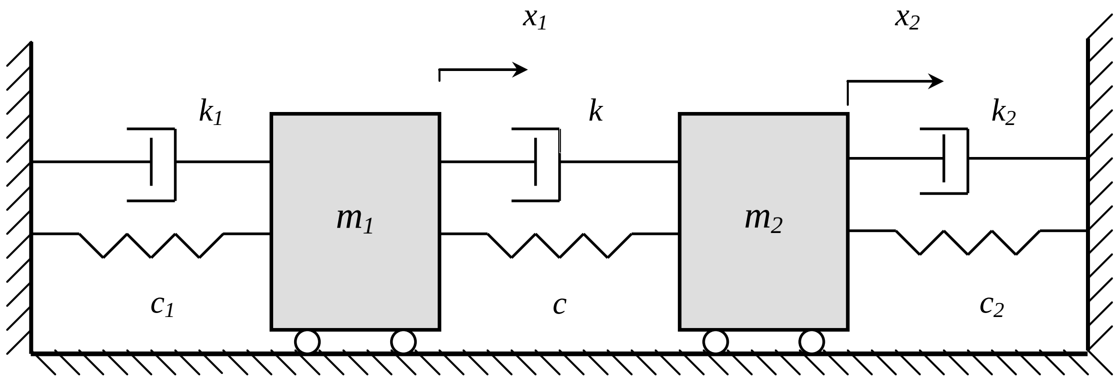


Figure 1. Two mass mechanical system.

The system (1) was object of intensive investigation from many authors [10-13] (and references there in). In [11], the necessary and sufficient condition for stability of equilibrium state $\bar{x}_e = 0$ (in linear case for spring and viscous damping coefficients) was obtained. In [13], the numerical analysis is performed by considering two Duffing oscillators, both with single-degree of freedom, coupled by a spring-dashpot system.

The aim of this paper is to elucidate how the stability of system (1) is affected by the nonlinear restoring forces represented in (2) and (3). In our qualitative analysis (in section 2) we use Routh-Hurwitz criterion for stability. Section 3 shows the results of our investigation through numerical simulations. Finally, Section 4 summarises our results.

2. Qualitative analysis

2.1. Symmetrical nonlinear restoring forces

Here, in our analytical approach, we consider one particular case (Eq. (2)) for restoring force, where $C = [c_1, c_2]^{-1}$ and $\Lambda = [\alpha_1, \alpha_2]^{-1}$ are matrices. Let us denote

(4) $$y_1 = x_1, \quad y_2 = \dot{x}_1, \quad y_3 = x_2, \quad y_4 = \dot{x}_2 .$$

After substitution of (4) and (2) into (1) and accomplishing some transformations, the system (1) is reduced to the four first-order differential equations:

(5) $$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -a_1 y_1 - a_2 y_2 + a_3 y_3 + a_4 y_4 - a_5 y_1^3, \\ \dot{y}_3 &= y_4, \\ \dot{y}_4 &= a_6 y_1 + a_7 y_2 - a_8 y_3 - a_9 y_4 - a_{10} y_3^3, \end{aligned}$$

where

(6) $$\begin{aligned} &a_1 = \frac{c + c_1}{m_1}, a_2 = \frac{k + k_1}{m_1}, a_3 = \frac{c}{m_1}, a_4 = \frac{k}{m_1}, a_5 = \frac{\alpha_1}{m_1}, \\ &a_6 = \frac{c}{m_2}, a_7 = \frac{k}{m_2}, a_8 = \frac{c + c_2}{m_2}, a_9 = \frac{k + k_2}{m_2}, a_{10} = \frac{\alpha_2}{m_2}. \end{aligned}$$

The steady (fixed points in the phase space) states of the system (5), $E = \left(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4 \right)$, are founded by equating the right-hand sides of (5) to zero. Thus, they can be analytically estimated and are defined by following set of algebraic equations, including the constants of the model:

(7) $$\bar{y}_1^{(1)} = \bar{y}_2^{(1)} = \bar{y}_3^{(1)} = \bar{y}_4^{(1)} = 0,$$

and

(8) $$\begin{aligned} &z^4 + K_3 z^3 + K_2 z^2 + K_1 z + K_0 = 0, \quad \bar{y}_2^{(2,3,4,5)} = \bar{y}_4^{(2,3,4,5)} = 0, \\ &\bar{y}_3^{2,3,4,5} = \frac{\bar{y}_1^{(2,3,4,5)}}{a_3} \left(a_1 + a_5 \left(\bar{y}_1^{(2,3,4,5)} \right)^2 \right), \end{aligned}$$

where

(9)
$$z=\bar{y}_1^{\,2},\; K_2=\frac{3a_1^2}{a_5^2},\; K_0=\frac{a_3^2}{a_5^3a_{10}}\left(a_1a_8-a_3a_6\right),$$
$$K_3=\frac{3a_1}{a_5},\; K_1=\frac{1}{a_5^3a_{10}}\left(a_3^2a_5a_8+a_1^3a_{10}\right).$$

In order to investigate the character of fixed points (7) and (8) we make the following substitutions into (5)

(10)
$$y_1=\bar{y}_1^{(j)}+w_1,\quad y_2=\bar{y}_2^{(j)}+w_2=w_2,$$
$$y_3=\bar{y}_3^{(j)}+w_3,\quad y_4=\bar{y}_4^{(j)}+w_4=w_4\quad \left(j=1-5\right).$$

Then, after accomplishing some transformations the system (5) has the form

(11)
$$\dot{w}_1=w_2,$$
$$\dot{w}_2=-a_{11}w_1-a_2w_2+a_3w_3+a_4w_4-a_{12}w_1^2-a_5w_1^3,$$
$$\dot{w}_3=w_4,$$
$$\dot{w}_4=a_6w_1+a_7w_2-a_{13}w_3-a_9w_4-a_{14}w_3^2-a_{10}w_3^3,$$

where

(12)
$$a_{11}=a_1+3a_5\left(\bar{y}_1^{(j)}\right)^2,\quad a_{12}=3a_5\,\bar{y}_1^{(j)},$$
$$a_{13}=a_8+3a_{10}\left(\bar{y}_3^{(j)}\right)^2,\quad a_{14}=3a_{10}\,\bar{y}_3^{(j)}.$$

The divergence of the flow (11) is

(13)
$$D_4=\sum_{l=1}^{4}\frac{\partial \dot{w}_l}{\partial w_l}=-a_2-a_9=-\left(\frac{k+k_1}{m_1}+\frac{k+k_2}{m_2}\right)<0.$$

Therefore, the system (11) is dissipative and has an attractor.
According to [14], the Routh-Hurwitz conditions for stability of (7) and (8) can be written in the form

(14)
$$p=a_2+a_9>0,$$

(15)
$$q=a_{11}+a_{13}+a_2a_9-a_4a_7=$$
$$=\frac{k\left(k_1+k_2\right)+k_1k_2}{m_1m_2}+\frac{c+c_1}{m_1}+\frac{c+c_2}{m_2}+3\left[\frac{\alpha_1}{m_1}L_1+\frac{\alpha_2}{m_2}L_2\right]>0,$$

(16)

$$r = a_2 a_{13} + a_9 a_{11} - a_4 a_6 =$$
$$= \frac{1}{m_1 m_2}\{kc_2 + k_1(c+c_2) + (c+c_1)(k+k_2) + 3[\alpha_1(k+k_2)L_1 + \alpha_2(k+k_1)L_2]\} > 0,$$

(17)
$$s = a_{11}a_{13} - a_3 a_6 =$$
$$= \frac{1}{m_1 m_2}\{c(c_1+c_2) + c_1 c_2 + 3[\alpha_1(c+c_2)L_1 + \alpha_2(c+c_1)L_2 + 3\alpha_1\alpha_2 L_1 L_2]\} > 0,$$

(18)

$$R = pqr - sp^2 - r^2 =$$
$$= a_2 a_9[(a_{11}+a_{13})(a_2 a_9 + a_4 a_7) + a_{11}^2 + a_{13}^2 + 2a_2^2 a_{13}(a_{11}a_{13} - a_3 a_6) + a_9(a_2 a_{13} - a_4 a_6) + a_9^2 a_{11}] +$$
$$+ a_4 a_6[(a_{11} - a_{13})(a_9 - a_2) + a_4 a_7 a_9 - a_4 a_6] + a_2\{a_2 a_3 a_6 - a_4[a_2(a_6 a_9 + a_7 a_{13}) + a_4 a_6 a_7]\} > 0,$$

where $L_1 = \left(\bar{y}_1^{(j)}\right)^2$ and $L_2 = \left(\bar{y}_3^{(j)}\right)^2$.

It is seen that Routh-Hurwitz conditions for stability (14)-(18) are always valid when

(19) $$\alpha_1 \geq 0, \quad \alpha_2 \geq 0, \quad k = 0.$$

Hence, we define the following theorem for stability of the system (5)

Theorem 1.

The necessary and sufficient conditions for asymptotically stability of the equilibriums (7) and (8) of system (5) are $k = 0, \quad \alpha_1 \geq 0$ and $\alpha_2 \geq 0$.

2.2. Unsymmetrical nonlinear restoring forces

Now, we consider the particular case when the restoring force F_R has the form (3) and $B = [\beta_1, \beta_2]^{-1}$ is a matrix. Thus, for original system (1) we have

(20)
$$\dot{y}_1 = y_2,$$
$$\dot{y}_2 = -a_1 y_1 - a_2 y_2 + a_3 y_3 + a_4 y_4 - a_{15} y_1^2,$$
$$\dot{y}_3 = y_4,$$
$$\dot{y}_4 = a_6 y_1 + a_7 y_2 - a_8 y_3 - a_9 y_4 - a_{16} y_3^2,$$

where $a_{15} = \frac{\beta_1}{m_1}$ and $a_{16} = \frac{\beta_2}{m_2}$. The fixed points of the system (20) are

(21) $$\bar{y}_1^{(1)} = \bar{y}_2^{(1)} = \bar{y}_3^{(1)} = \bar{y}_4^{(1)} = 0,$$

(22) $$\bar{y}_1^{3} + M_2 \bar{y}_1^{2} + M_1 \bar{y}_1 + M_0 = 0, \quad \bar{y}_3 = \frac{\bar{y}_1^{(2,3,4)}}{a_3}\left(a_1 + a_5 \bar{y}_1^{(2,3,4)}\right),$$

where $M_2 = \frac{2a_1}{a_{15}}, M_1 = \frac{1}{a_{15}^2}\left(a_1^2 + \frac{a_3 a_{15} a_8}{a_{16}}\right)$ and $M_0 = \frac{a_3}{a_{15}^2 a_{16}}\left(a_1 a_8 - a_3 a_6\right)$. According to the Descarte's rule [15], the first equation in (22) has (i) always one real root; (ii) two real roots or (iii) three different real roots, which are physically feasible. Following the same type of procedure (see section 3.1), it is seen that the Routh-Hurwitz conditions for stability of fixed points (21) and (22) are those as (14)-(18). Note that now $a_{11} = a_1 + 2a_{15}\bar{y}_1^{(j)}$, $a_{13} = a_8 + 2a_{16}\bar{y}_3^{(j)}$, $L_1 = \bar{y}_1^{(j)}$, $L_2 = \bar{y}_3^{(j)}$ and $(j = 1-4)$. Thus, we define the following Theorem 2 for stability of system (20):

Theorem 2.

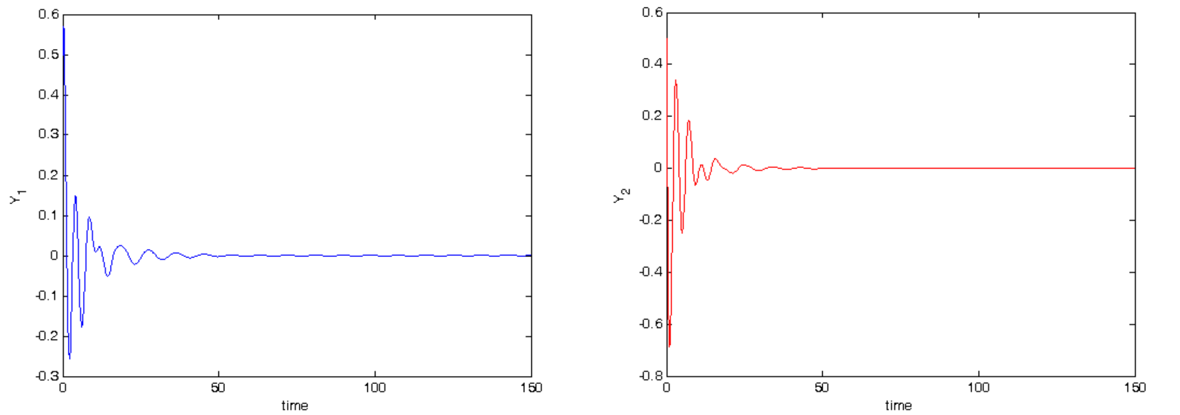
The necessary and sufficient conditions for asymptotically stability of the equilibriums (21) and (22) of system (20) are $k = 0$, $\beta_1 \geq 0$, $\beta_2 \geq 0$; L_1 *and* L_2 *to be real non-negative numbers.*

3. Numerical analysis

In previous section, we introduced the analytical tools proposed and used them for a qualitative analysis of the system (1) obtaining some predictions about the stability behavior of this system. The values chosen for the parameters and used in numerical analysis are:

(23) $$\begin{aligned} &m_1 = 4\,[kg], \quad m_2 = 35[kg], \quad k = 10[Ns/m], \quad k_1 = 10[Ns/m], \\ &k_2 = 50[Ns/m], \quad c = 40[N/m], \quad c_1 = 50[N/m], \quad c_2 = 160[N/m], \\ &\alpha_1 \in [-200, 200][N/m^3], \quad \alpha_2 \in [-200, 200][N/m^3], \quad \beta_1 \in [-200, 200][N/m^2], \\ &\beta_2 \in [-200, 200][N/m^2] \end{aligned}$$

In Figure 2, we illustrate the numerical computations of system (5), when m_1=4, m_2=35, c_1=50, c_2=160, c=40, k=0, k_1=10; k_2=50, l_1=135 and l_2=200. In this case the Routh-Hurwitz conditions (14)-(18) for stability are valid (and Theorem 1 also), and the system (5) has only stable solutions.



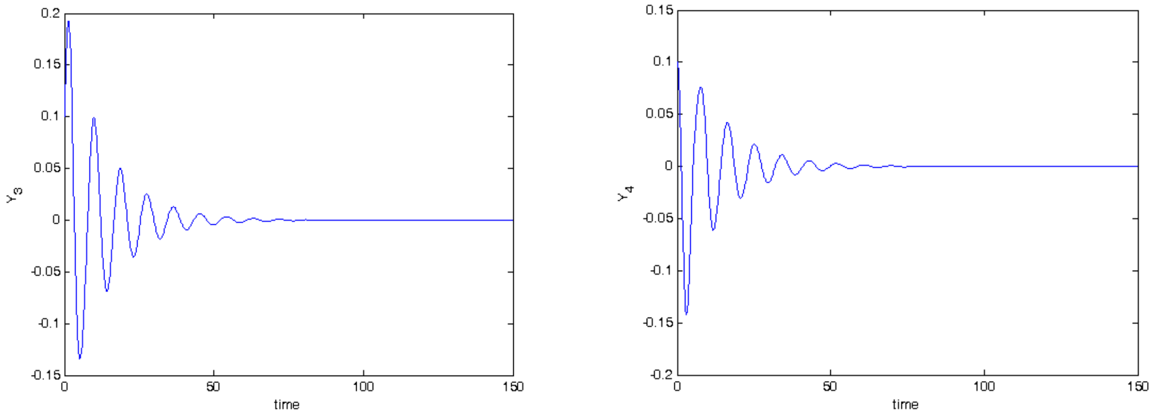


Figure 2. Stability solutions of system (5) when the conditions of Theorem 1 are valid, i.e. m_1=4; m_2=35; c_1=50; c_2=160; c=40; k=0; k_1=10; k_2=50; l_1=135; l_2=200.

In our numerical simulations depicted in Figure 3, it is seen that for $l_1 = -138$ the system (5) has unstable solutions. Note that in this case the conditions for stability in Theorem 1 are not valid.

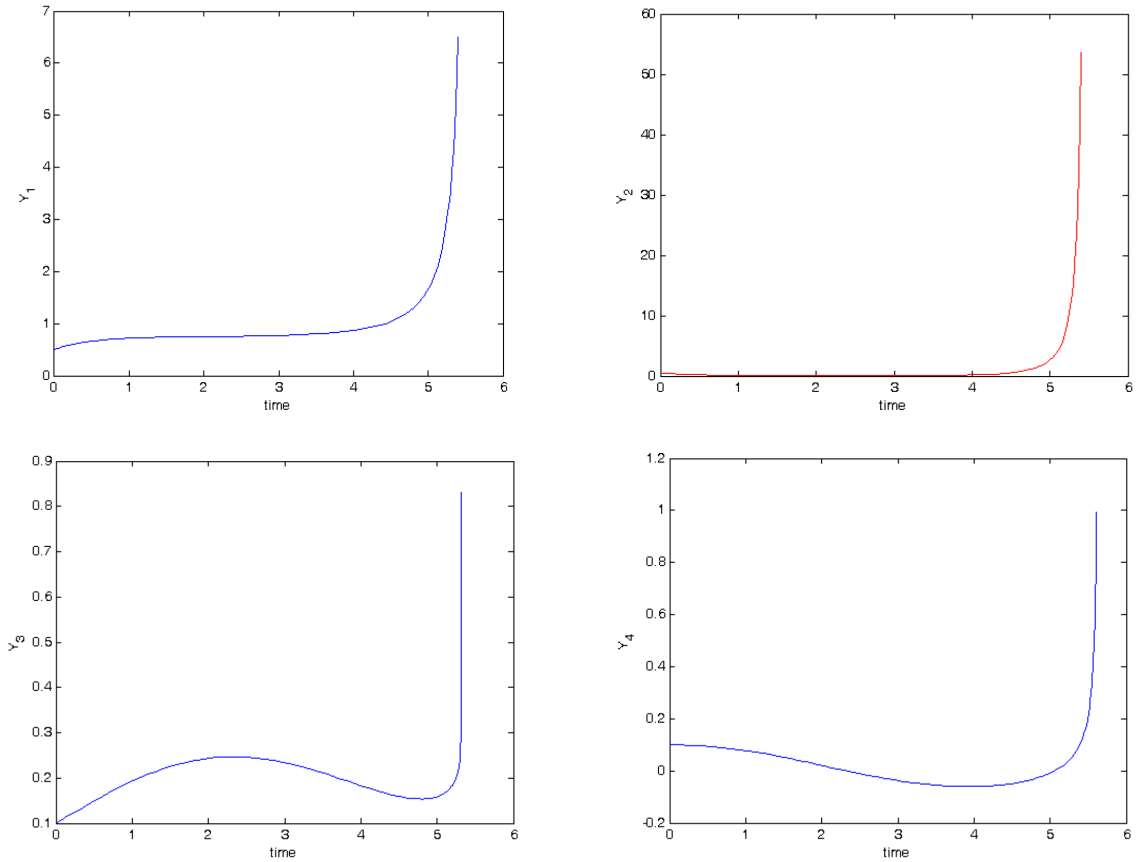


Figure 3. Unstable solutions of system (5) when conditions of Theorem 1 are not valid, i.e. l_1=-138.

4. Conclusions

The paper presents a study of the stability behavior of a two mass nonlinear mechanical system, using Routh-Hurwitz criterion for stability. Considering the cases of symmetrical and unsymmetrical nonlinear restoring force, we find necessary and sufficient condition for stability of system (1). In Section 3, we check the validity of our analytical results with a numerical example. Note that these results are in accordance with the analytical results obtained in Section 2. Generalizing our results in Section 2 and 3, we can conclude that nonlinear fragment (in restoring force) acts as a key parameter in stability behaviour of system (1).

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