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DYNAMICS OF A DOUBLE PENDULUM

Svetoslav Nikolov[1,2]
S.Nikolov@imbm.bas.bg

[1]University of Transport, G. Milev Str. No 158, 1574 Sofia,
[2]Institute of Mechanics-Bulgarian Academy of Sciences,
Acad. G. Bonchev Str., Bl. 4 1113 Sofia,
THE REPUBLIC OF BULGARIA

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Abstract: *In this paper we consider the motion of the double pendulum attached to a platform that has a prescribed vertical oscillation relative to an inertial frame in Hamiltonian context. In order to investigate the dynamics of the system, we obtain its Hamiltonian which has unperturbed and perturbed parts. Thus, some analytical and numerical results about dynamics of the system are obtained.*

1. INTRODUCTION

The pendulum systems are classical models of nonlinear dynamics [1]. They have numerous applications in many physical processes and engineering [2-6]. It is well-known that Hamiltonian systems with two (and more) degrees of freedom have interesting applications and can exhibit trajectories with complicated behavior [7,8]. Note that, systems called Hamiltonian if their equations can be written in canonical form by means of a Hamiltonian $H(q,p,t)$, where q and p are generalized coordinates and momenta, and t is the time.

Integrability and stability of Hamiltonian systems are central problems. The integrability is associated with averaging methods [9].

Consider the averaging (method) [10] in the Hamiltonian systems. Thus

(1) $$H(q,p,\varepsilon)=H_0(p)+\varepsilon H_1(q,p,\varepsilon),$$

where $H_0(p)$ is the integrable unperturbed Hamiltonian, $H_1(q,p,\varepsilon)$ is the perturbed Hamiltonian which is 2π - periodic in components of q (fast variables), and ε is a small parameter (i.e. $\varepsilon << 1$) characterising the magnitude of the perturbation. The equations of motion are

(2) $$\dot{p}=-\frac{\partial H(q,p,\varepsilon)}{\partial q}=-\varepsilon\frac{\partial H_1}{\partial q}, \quad \dot{q}=\frac{\partial H(q,p,\varepsilon)}{\partial p}=\frac{\partial H_0(p)}{\partial p}+\varepsilon\frac{\partial H_1}{\partial q}.$$

Using averaging method for the perturbed Hamiltonian H_1, we obtain the averaged Hamiltonian in the form

(3) $$\widetilde{H}(p,\varepsilon)=H_0(p)+\varepsilon\widetilde{H}_1(p),$$

where $\widetilde{H}_1=\frac{1}{(2\pi)^n}\int_{[0,2\pi)^n}H_1(q,\,p,\,0)dq$. Here n is the number of degrees of freedom.

It follows from the Nekhoroshev's theorem [11] that, if $H_0(p)$ is a steep function, then there exist positive constants a,b and c such that for the perturbed Hamiltonian system (for a sufficiently small perturbation) we can suppose the form

(4) $$|p(t)-p(0)|<\varepsilon^b \quad , \quad 0\le t\le \left(1/\varepsilon\right)\ell^{\frac{c^{-1}}{\varepsilon^a}}.$$

The Hamiltonian system associated to $H_0(p)$ is integrable.

This paper is concerned to complete the study of the dynamical aspects of a mechanical system (double pendulum) with two and a half degrees of freedom.

2. HAMILTONIAN FORM OF EQUATIONS OF MOTION

We consider the motion of a double pendulum attached to a platform that has a prescribed vertical motion relative to an inertial frame – see Figure 1. According to [1,8], the

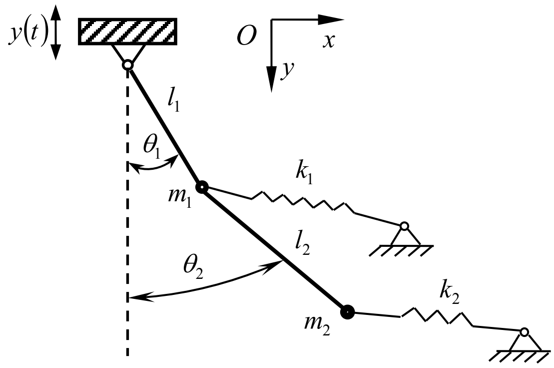

Figure 1. A double pendulum with a moving support.

kinetic energy T and the potential energy U of this system have the form

(5)
$$T=\frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2-(m_1+m_2)l_1\dot{y}\dot{\theta}_1\theta_1+\frac{1}{2}m_2l_2^2\dot{\theta}_2^2+m_2l_1l_2\dot{\theta}_1\dot{\theta}_2-$$
$$-m_2l_2\dot{y}\dot{\theta}_2\theta_2+\frac{1}{2}(m_1+m_2)\dot{y}^2+P_1(\theta_j^3),$$
$$U=-m_1g(y+l_1\cos\theta_1)-m_2g(y+l_1\cos\theta_1+l_2\cos\theta_2)+\frac{1}{2}k_1l_1^2\theta_1^2+$$
$$+\frac{1}{2}k_2(l_1\theta_1+l_2\theta_2)^2+P_2(\theta_j^3)\qquad (j=1,2),$$

where m_1 and m_2 are particles masses, l_1 and l_2 are the lengths of massless rods, k_1 and k_2 are the springs constants, g is the acceleration of the gravity and $y(t)$ is the vertical motion of the platform with respect to the inertial frame O. It is assumed that $y(t)$ is a periodic fast oscillating function of time. Note that the springs are unstretched when the particles lie vertically below the platform.

To obtain the Hamiltonian $H(\theta_1, p_1, \theta_2, p_2)$ of the pendulum, first we pass to the canonical momentum representation

(6) $$p_1 = \frac{\partial T}{\partial \dot{\theta}_1} \quad , \quad p_2 = \frac{\partial T}{\partial \dot{\theta}_2} .$$

Thus the Hamiltonian is

(7) $$\begin{aligned} H(\theta_1, p_1, \theta_2, p_2) = T + U = c_1 p_1^2 + c_2 p_2^2 - c_3 p_1 p_2 + c_4 \theta_1^2 + c_5 \theta_2^2 + \\ + c_6 \theta_1 \theta_2 - c_7 \cos\theta_1 - c_8 \cos\theta_2 + a_0 + a_1 \theta_1 p_2 - a_2 \theta_1^2 - a_3 \theta_2^2 + a_4 \theta_1 \theta_2 , \end{aligned}$$

where

(8) $$\begin{aligned} & c_1 = \frac{1}{2 m_1 l_1^2}, \quad c_2 = \frac{(m_1 + m_2)(1 + m_1 - m_2)}{2 m_1^2 m_2 l_2^2}, \quad c_3 = \frac{1}{m_1 l_1 l_2}, \quad c_4 = \frac{1}{2}(k_1 + k_2) l_1^2 , \\ & c_5 = \frac{1}{2} k_2 l_2^2 , \quad c_6 = k_2 l_1 l_2 , \quad c_7 = g l_1 (m_1 + m_2), \quad c_8 = m_2 g l_2 , \\ & a_0 = (m_1 + m_2)\left(\frac{1}{2}\dot{y}^2 - g y\right), \quad a_1 = \frac{(m_1 + m_2)(m_1 + m_2 - 1)\dot{y}}{m_1^2 l_2}, \quad a_2 = \frac{(m_1 + m_2)^2 \dot{y}^2}{2 m_1} , \\ & a_3 = \frac{m_2 (m_1 + m_2) \dot{y}^2}{2 m_1}, \quad a_4 = \frac{m_2 (m_1 + m_2) \dot{y}^2}{m_1} . \end{aligned}$$

Then there exist the governed canonical equations of motion in the form

(9) $$\begin{aligned} & \dot{\theta}_1 = \frac{\partial H}{\partial p_1} = \frac{1}{m_1 l_1^2} p_1 - c_3 p_2 + \frac{(m_1 + m_2)\dot{y}}{m_1 l_1} \theta_1 - \frac{m_2 \dot{y}}{m_1 l_1} \theta_2 , \\ & \dot{\theta}_2 = \frac{\partial H}{\partial p_2} = -c_3 p_1 + \frac{m_1 + m_2}{m_1 m_2 l_2^2} p_2 + \frac{(m_1 + m_2)\dot{y}}{m_1 l_2} (\theta_2 - \theta_1), \\ & \dot{p}_1 = -\frac{\partial H}{\partial \theta_1} = \frac{(m_1 + m_2)\dot{y}}{m_1 l_2} p_2 + \left[\frac{m_2 (m_1 + m_2)\dot{y}^2}{m_1} - (k_1 + k_2) l_1^2\right]\theta_1 - k_2 l_1 l_2 \theta_2 + g(m_1 l_1 + m_2 l_2)\sin\theta_1 , \\ & \dot{p}_2 = -\frac{\partial H}{\partial \theta_2} = -\frac{2(m_1 + m_2)\dot{y}}{m_1 l_2} p_2 - \left[\frac{m_2 (m_1 + m_2)\dot{y}^2}{m_1} + k_2 l_1 l_2\right]\theta_1 + \left[\frac{m_2 (m_1 + m_2)\dot{y}^2}{m_1} - k_2 l_2^2\right]\theta_2 - \\ & \qquad - m_2 g l_2 \sin\theta_2 . \end{aligned}$$

3. AVERAGED HAMILTONIAN

Assume that $y = \varepsilon \tilde{y}\left(t/\varepsilon\right)$, where $\tilde{y}(\cdot)$ is a 2π-periodic function of the argument t/ε with a zero average. Then $\dot{y}$ is value of order 1. According to [10], we average the Hamiltonian H with respect to time t.

For $H_1 \equiv 0$ (see (1)), $H = H_0$, where the unperturbated Hamiltonian (which is very rapidly oscillating in time) has the form

(10) $$H_0(p_1, p_2) = c_1 p_1^2 + c_2 p_2^2 - c_3 p_1 p_2 .$$

In fact, the phase portrait of the system will be foliated by the curves of the Hamiltonian H_0. Indeed, the phase space of the unperturbed system is foliated by invariant two-dimensional (resonant/nonresonant) tori $N^2_{p_0} = \{(\theta, p): p = p_0 = const.\}$ with frequences $\upsilon(p) = \dfrac{\partial H_0}{\partial p}$ and filled with quasi-periodic solutions: $\theta = \upsilon(p)t + \theta_0$, where $p \in R^2$ and $\theta = (\theta_1, \theta_2)$ - see Fig. 2. The unperturbed system is

(11)
$$\dot{\theta}_1 = p_1 - p_2\,, \quad \dot{\theta}_2 = -p_1 + 2p_2\,,$$
$$\dot{p}_1 = -2\theta_1 - \theta_2\,, \quad \dot{p}_2 = -\theta_1 - \theta_2\,,$$

where without loss of generality we assume that $m_1 = m_2 = 1[kg], l_1 = l_2 = 1[m]$ and $k_1 = k_2 = 1\left[N/m\right]$.

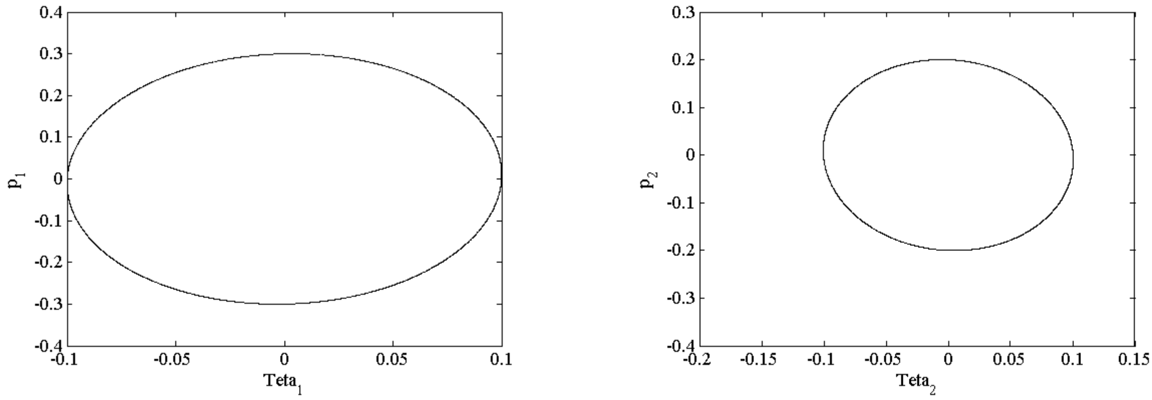

Figure 2. Phase portait of system (11) for initial conditions: $\theta_{10} = \theta_{20} = 0.1$, $p_{10} = p_{20} = 0.01$.

For $H_1 \neq 0$ and $0 < |\varepsilon| << 1$, the dynamics of system (9) becomes more intricate. Hence, the averaged Hamiltonian is

(12)
$$\widetilde{H} = H_0 + \widetilde{U}(\theta_1, \theta_2) = H_0 + c_4\theta_1^2 + c_5\theta_2^2 + c_6\theta_1\theta_2 -$$
$$-\frac{m_1 + m_2}{m_1}\left[(m_1 + m_2)\theta_1^2 + m_2\theta_2^2 - 2m_2\theta_1\theta_2\right]\Psi - c_7\cos\theta_1 - c_8\cos\theta_2\,,$$

where Ψ denote the average of $\dfrac{(\dot{y})^2}{2}$. It is essential to note that the type of the phase portrait of H depends on the extrema of $\widetilde{U}$. In fact, the function $\widetilde{U}$ depends on Ψ. On the other hand, $\widetilde{U}$ is invariant under the transformation $\theta_1 \to -\theta_1$, $\theta_2 \to -\theta_2$. Below we consider two cases: i) $\Psi = 0$ and ii) $\Psi \neq 0$.

Special case $\Psi = 0$

The system is autonomous and the conservation of energy $(\dot{H} = 0)$ takes place. The equilibrium (fixed) points of the system are

(13)
$$p_1 = p_2 = 0\,, \quad \theta_1 = \theta_2 = k\pi \quad (k = 0, \pm 1, \pm 2, \ldots).$$

According to [12], the law of energy conservation $E = T + U$ defines an invariant three-dimensional hypersurface $H = E$ as a special phase curve- homoclinic orbit, lying in the three-dimensional manifold with level of energy E_h, exists.

Case $\Psi \neq 0$

We get the equilibrium $P_0(\theta_1,\theta_2)=P_0(0,0)$. For this point $\left(\frac{\partial\widetilde{U}}{\partial\theta_1}\right)_0=0$ and $\left(\frac{\partial\widetilde{U}}{\partial\theta_2}\right)_0=0$.

Thus, the function $\widetilde{U}(\theta_1,\theta_2)$ has extrema in this point if and only if

(14) $$F(\Psi)=B^2-AC<0,$$

where : for $A<0, C<0$ - maximum; for $A>0, C>0$ - minimum. In (14)

(15) $$A=\left(\frac{\partial^2\widetilde{U}}{\partial\theta_1^2}\right)_0, \quad B=\left(\frac{\partial^2\widetilde{U}}{\partial\theta_1\partial\theta_2}\right)_0, \quad C=\left(\frac{\partial^2\widetilde{U}}{\partial\theta_2^2}\right)_0.$$

Here, for A, B and C we have

(16) $$A=2c_4-\frac{2(m_1+m_2)^2}{m_1}\Psi+c_7, \quad B=c_6+\frac{2m_2(m_1+m_2)}{m_1}\Psi, \quad C=2c_5-\frac{2m_2(m_1+m_2)}{m_1}\Psi+c_8.$$

For $m_1=m_2=1[kg], l_1=l_2=1[m]$, $g=10\left[m/_{s^2}\right]$ and $k_1=k_2=1\left[N/_m\right]$, $F(\Psi)<0$ when $\Psi\in(-\infty,1.5074)\cup(9.9926,+\infty)$. Thus, after a direct calculation (see Fig. 3) we obtain that: i) if $\Psi\in(-\infty,1.5074)$, then $A>0, C>0$ and the equilibrium $P_0(\theta_1,\theta_2)=P_0(0,0)$ is a minimum; if $\Psi\in(9.9926,+\infty)$, then $A<0, C<0$ and the equilibrium $P_0(\theta_1,\theta_2)=P_0(0,0)$ is a maximum.

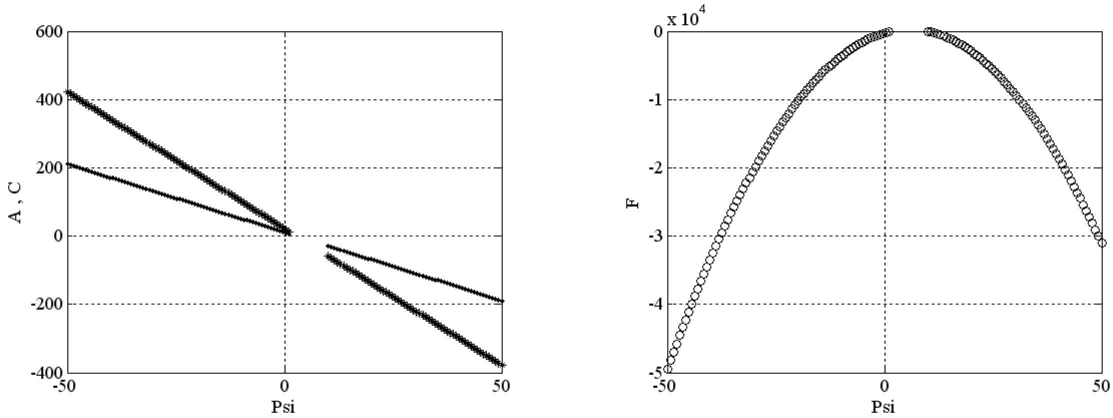

Figure 3. Values of *A* (stars) and *C* (points) (a), and F (b), when $m_1=m_2=1[kg], l_1=l_2=1[m]$, $g=10\left[m/_{s^2}\right]$, $k_1=k_2=1\left[N/_m\right]$ and $\Psi\in(-\infty,1.5074)\cup(9.9926,+\infty)$

4. CONCLUSION

In this paper a double pendulum whose suspension point performs high-frequency vibranions was considered. The system has $2\frac{1}{2}$ degres of freedom with canonical variables θ_1,θ_2,p_1 and p_2. After averaging with respect to phase of fast oscillations we obtain a system with two degrees of freedom. Value of the unperturbed Hamiltonian H_0 (when $H_1\equiv 0$) is a first integral of the averaged system. Our study of the averaged system provides useful information about dynamics of the original (not averaged) model. Moreover, our findings here should stimulate further analytical and numerical studies of system (9).

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ДИНАМИКА НА ЕДНО ДВОЙНО МАХАЛО

Светослав Николов[1,2]
S.Nikolov@imbm.bas.bg

[1]Висше транспортно училище „Т. Каблешков”, ул. Г. Милев No 158, 1574 София,
[2]Институт по Механика-Българска Академия на Науките,
ул. Акад. Г. Бончев, бл. 4, 1113 София,
РЕПУБЛИКА БЪЛГАРИЯ

Ключови думи: *двойно махало, анализ, неавтономен Хамилтониан*

Резюме: *В тази статия (в Хамилтонов контекст) разглеждаме движението на двойно махало, което е прикачено за платформа извършваща вертикално трептене относно относителна инерционна координатна система. За да изследваме динамичното поведение на системата добиваме нейния Хамилтониан, който има несмутена и смутена част. По този начин, се получават редица аналитични и числени резултати разкриващи особеностите в динамиката на системата.*