Mechanics Transport Communications
Academic journal http://www.mtc-aj.com

ISSN 1312-3823 (print)
ISSN 2367-6620 (online)
volume 19, issue 3, 2021
article № 2119

EFFECT OF SYNCHRONIZATION ON A SYSTEM OF HOPF-LANGFORD TYPE

Svetoslav G. Nikolov[1,2]
S.Nikolov@imbm.bas.bg

[1]University of Transport, Geo Milev Str., 158, 1574 Sofia,
[2]Institute of Mechanics-Bulgarian Academy of Sciences,
Acad. G. Bonchev Str., bl. 4, 1113 Sofia,
THE REPUBLIC OF BULGARIA

Key words: *analysis, synchronization, nonlinear dynamics, Hopf-Langford system*

Abstract: *In this paper, the dynamic behavior of a 3D autonomous dissipative nonlinear system of Hopf-Langford type is investigated qualitatively and numerically. It is shown that the 3D nonlinear system can be separated of two coupled subsystems in the master (drive)-slave (response) synchronization type if the system's energy is included. Based on the computing first Lyapunov value for master system, we have attempted to give a general framework (from bifurcation theory point of view) for understanding the structural stability and bifurcation behavior of original system. The effect of synchronization on the dynamic behavior of original system is also studied by numerical simulations.*

1. INTRODUCTION

Chaotic behavior has been observed in systems of different nature as this motion is based on homoclinic (heteroclinic) structures which instability accompanied by local divergence and global contraction [1-3]. It is well-known that autonomous nonlinear differential system of the form

$$\dot{x} = \frac{dx}{dt} = f(x, \lambda), \quad x \in R^n ,$$

where $n \geq 3$, λ is the vector of parameters and $f : R^n \to R^n$ is a smooth vector function (i.e. continuously differentiable) in some domain Ω, can display a rich diversity of periodic, multiple periodic, chaotic and hyperchaotic flows dependent upon the specific values of one or more bifurcation (control) parameters [4, 5].

The investigation of dynamical processes in coupled nonlinear systems is an interesting problem from both theoretical (mathematical) and applied (engineering) points of view. Phenomena such as stability in interacting subsystems can be observed in nature and science. Usually, that phenomena is called synchronization [6, 7]. There are known four basic types of synchronization: complete, generalized, phase and lag synchronization [8]. Phase synchronization is the phenomenon of the onset of balance between the phases of the subsystems state variables oscillations, which is caused by an onset of the energy balance.

A principal problem toward complete understanding of nonlinear interactions is to identify where in its phase space one dynamical system is structurally stable. For example, in a small neighborhood of a structurally stable Poincare homoclinic orbit lie only periodic orbits from saddle type. On the contrary, near a structurally unstable homoclinic orbit may exist both structurally unstable and attractive periodic orbits in addition to saddle ones [9]. Note that after Smale's works [10, 11] these systems are said to be Morse-Smale systems.

The structural stability (roughness) investigation of steady state and of limit cycles or other types of trajectories is a main problem in bifurcation theory. It is well-known that there is critical dependence of the stability conditions of limit cycles on the stability conditions of its steady states. Based on classical works [12-14], it was defined that by knowing the sign of Lyapunov values (called also focus values, Lyapunov quantities (coefficients)) we can efficiently studied the structure of complicated nonlinear system trajectories. In other words, the type of: 1) stability loss of equilibrium and 2) winding/unwinding of system trajectories in small neighborhoods of equilibrium depend on the sign of Lyapunov value [15, 16].

In this paper, we focus our study on the following system

(1)
$$\begin{aligned} \dot{x}_1 &= (\mu-\alpha)x_1 - \beta x_2 + x_1x_3 + lx_1\left(1-x_3^2\right), \\ \dot{x}_2 &= \beta x_1 + (\mu-\alpha)x_2 + x_2x_3 + lx_2\left(1-x_3^2\right), \\ \dot{x}_3 &= \mu x_3 - \gamma\left(x_1^2+x_2^2+x_3^2\right), \end{aligned}$$

where $\mu, \alpha, \beta, \gamma$ and l are the positive system parameters. The infinite form from ordinary differential equations of system (1) was originally introduced by Hopf [17] in order to describe a fluid turbulence dynamics. Later, firstly in a private communication and after that in a paper [18], Langford constructed (1) for $\alpha = \beta = \gamma = 1$ and $l = 0$. In our next considerations, when $\alpha \neq 1, \beta \neq 1, \gamma = 1$ and $l \neq 0$, we will called system (1) – generalized Hopf-Langford system (GHL). After the works of Hassard et al. [19] and Nikolov et al. [20], even today, the GHL system represents an attractive example for both theoretical and numerical investigations [21-23].

Here we investigate the previously unexplored parameter regions of the generalized Hopf-Langford system (GHL). A new qualitative picture of behavior near bifurcation points can be obtained if the GHL system (1) has the following modified form

(2)
$$\begin{aligned} \dot{x}_1 &= (\mu-\alpha)x_1 - \beta x_2 + x_1x_3 + lx_1\left(1-x_3^2\right), \\ \dot{x}_2 &= \beta x_1 + (\mu-\alpha)x_2 + x_2x_3 + lx_2\left(1-x_3^2\right), \\ \dot{x}_3 &= \mu x_3 - 2E, \\ \dot{E} &= 2(\mu+l-\alpha)E - (l-\alpha)x_3^2 - 2lx_3^2E - x_3^3 + lx_3^4, \end{aligned}$$

where $E = \frac{1}{2}\left(x_1^2+x_2^2+x_3^2\right)$ is the energy. Moreover, we investigate qualitatively and numerically the structural stability of (2) (which is equivalent of the original system (1)) as for this goal a specific version of bifurcation theory, based on the computing of Lyapunov values (not exponents), was used.

The fixed (steady state) points (FPs)of the system (2) are found by equating the right-hand sides of (2) to zero. Thus, we obtain that equilibrium points of the system are

(3) **O₁**: $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = \bar{E} = 0$, *first FP,*

(4) **O$_{2,3,4}$**: $\bar{x}_1 = \bar{x}_2 = 0, \quad \bar{E} = \frac{\mu}{2}\bar{x}_3,$ $\quad l\bar{x}_3^3 - (\mu l+1)\bar{x}_3^2 + (\alpha - l)\bar{x}_3 + \mu(\mu+l-\alpha) = 0.$ *second, third and fourth FPs.*

Here we note that the polynomial function for $\bar{x}_3$ in (4) has three different real roots if

$$Q = \left(\frac{K_1}{3}\right)^3 + \left(\frac{K_2}{2}\right)^2 < 0, \tag{5}$$

where $K_1 = -\frac{m_1^2}{3} + m_2$, $K_2 = 2\left(\frac{m_1}{3}\right)^3 - \frac{m_1 m_2}{3} + m_3$, $m_1 = -\frac{\mu l + 1}{l}$, $m_2 = \frac{\alpha - l}{l}$ and $m_3 = \frac{\mu(\mu + l - \alpha)}{l}$. While dealing with modified system (2), the main simplification is that it can be separated of two coupled subsystems in the master (drive) – slave (response) synchronization type. As we can see, the master system has the form

$$\begin{aligned} \dot{x}_3 &= \mu x_3 - 2E, \\ \dot{E} &= 2(\mu + l - \alpha)E - (l - \alpha)x_3^2 - 2lx_3^2 E - x_3^3 + lx_3^4. \end{aligned} \tag{6}$$

Since the corresponding dynamic behavior of the original system (2) principally dependence from the behavior of the master system [24], below we investigate only the bifurcation dynamic of this system. It is seen that master system describes the evolution in time of variable x_3 and energy E.

The plan of the paper is as follows: in Section 2 we present analytical and numerical results concerning the system (6). In Section 3 we summarize our results.

2. QUALITATIVE AND NUMERICAL ANALYSIS

In this section we consider system (6) which present an autonomous nonlinear dynamical model. According to the general theory of ordinary differential equations [25], the equilibrium (steady state) values of the system (6) are as those into (3) and (4), the part for x_3 and E. In order to determine the character of these fixed points, we make the following substitutions into (6)

$$x_3 = \bar{x}_3 + x, \quad E = \bar{E} + y. \tag{7}$$

Hence, after accomplishing some transformations, the system (6) (in local coordinates) can be written in the form

$$\begin{aligned} \dot{x} &= \mu x - 2y, \\ \dot{y} &= c_1 x + c_2 y + c_3 x^2 - c_4 xy + c_5 x^3 - c_6 x^2 y + lx^4, \end{aligned} \tag{8}$$

where

$$\begin{aligned} c_1 &= \bar{x}_3\left[4l\,\bar{x}_3^2 - 3\bar{x}_3 - 4l\bar{E} - 2(l - \alpha)\right],\ c_2 = 2\left(\mu + l - \alpha - l\,\bar{x}_3^2\right); \\ c_3 &= \left(6l\,\bar{x}_3^2 - 3\bar{x}_3 - 2l\bar{E} - l + \alpha\right),\ c_4 = 4l\,\bar{x}_3,\ c_5 = 4l\,\bar{x}_3 - 1,\ c_6 = 2l. \end{aligned} \tag{9}$$

The stability of fixed points (3) and (4) is defined by the following Routh-Hurwitz conditions

$$R \equiv p = -(\mu + c_2) = -3\mu + 2\left(l\,\bar{x}_3^2 + \alpha - l\right) > 0, \tag{10}$$

$$q = \mu c_2 + 2c_1 = 2\left\{\mu\left(\mu + l - \alpha - l\,\bar{x}_3^2\right) + \bar{x}_3\left[4l\,\bar{x}_3^2 - (3 + 2l\mu)\bar{x}_3 - 2(l - \alpha)\right]\right\} > 0. \tag{11}$$

The notations p, q and R in (10) and (11) are taken from [14]. It is seen that first fixed point O_1 (eq. (3)) is always stable if

$$\begin{cases} \alpha < \mu + l, \\ \alpha > \dfrac{2l + 3\mu}{2} \cdot \end{cases} \tag{12}$$

When the condition (10) is not valid, the steady states (3) and (4) become unstable, as in this case according to [14] "soft" (reversible) or "hard" (un-reversible) stability loss takes place. In order to define the type of stability loss of steady states (3) and (4) it is necessary to calculate the so-called first Lyapunov value ($L_1(\lambda_0)$) on the boundary of stability $R = 0$. In case of two first order differential equations, this value can by determined analytically by the formula in [14]. For the system (8) we have

$$\begin{aligned} &a_{11} = a_{02} = a_{20} = a_{30} = a_{21} = a_{12} = b_{02} = b_{03} = b_{12} = 0, a = \mu, \\ &b = -2, c = c_1, d = c_2, b_{20} = c_3, b_{11} = -c_4, b_{30} = c_5, b_{21} = -c_6. \end{aligned} \tag{13}$$

Thus, in our case we obtain for $L_1(\lambda_0)$:

$$L_1(\lambda_0) = \frac{\pi d}{2q\sqrt{q}}\left[16l\,\bar{x}_3^3 - 2(3 + 4\mu l)\bar{x}_3^2 + \mu^2\right]. \tag{14}$$

It is seen that: i) for $l = 0$ then $L_1 = 0$; (ii) for $l \neq 0$ and $\bar{x}_3 = 0$ then $L_1 = \dfrac{\pi d \mu^2}{2q\sqrt{q}} > 0$ and iii) for $l \neq 0, \bar{x}_3 \neq 0$ then L_1 can be positive, negative or equal to zero.

In order to simplify our numerical analysis, some of the parameters are kept constant. The parameter values are: $\alpha = 1$ and $\beta = 0.9$. The first Lyapunov value obtained for the system (6) as a function of bifurcation parameters $\mu \in [0.53, 0.533]$ and $l \in [0.208, 0.24]$ is plotted in Figure 1. It is seen that for smaller values of μ and l, L_1 has negative values, i.e. soft stability loss take place.

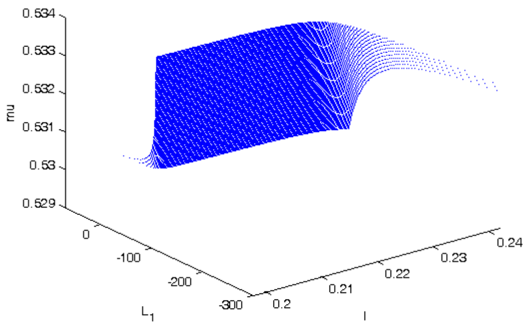

Figure 1. First Lyapunov value L_1 as function of parameters μ and l, when $\alpha = 1$.

In Figures 2 and 3, we numerically demonstrate that when master system (6) is stable (i.e. the energy E of system is a constant) or has a stable limit cycle then slave system has periodic solutions with period one, or quasi-periodic solutions (oscillations) with period different from one.

3. CONCLUSION

The nonlinear behavior (effect of synchronization) of a dissipative system of Hopf-Langford type has been investigated qualitatively and numerically. It has been shown that, if

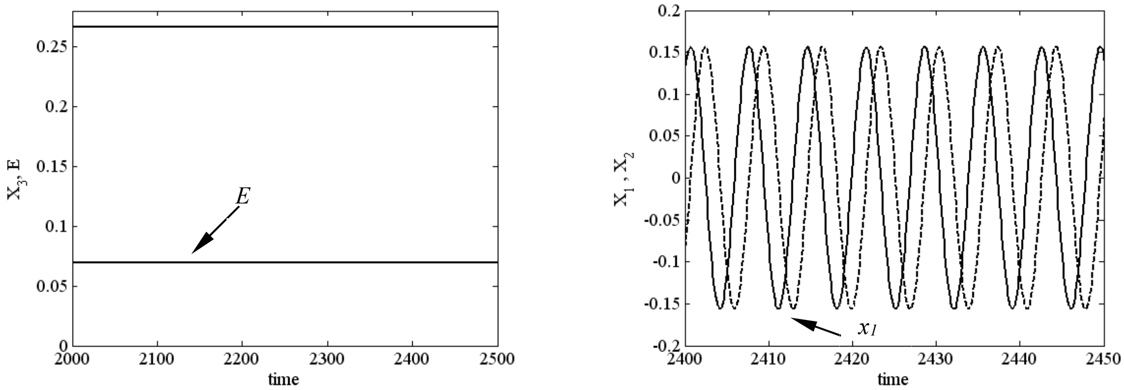

Figure 2. Periodic solutions for slave system, when energy *E* is constant.

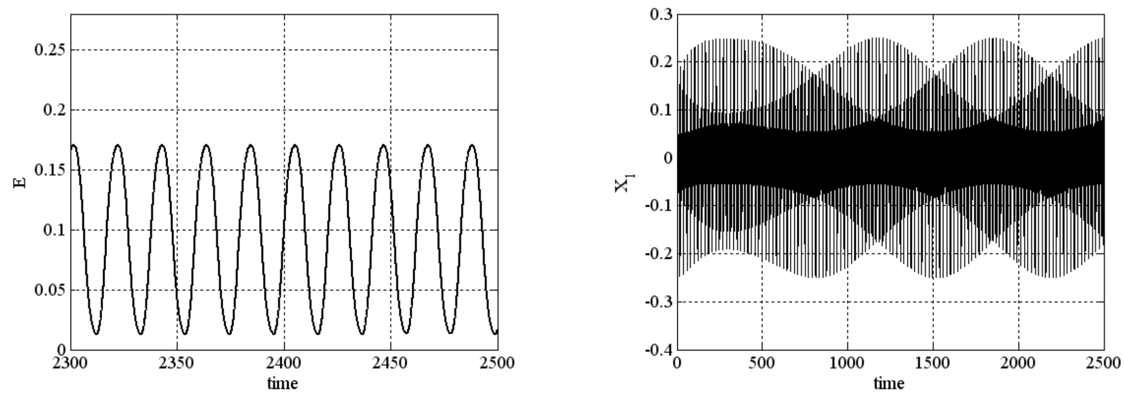

Figure 3. Quasi-periodic solutions of slave system, when energy *E* oscillate.

master system (energy) is constant then the slave system has periodic solutions. On the other hand, if the energy oscillate then the slave system has quasi-periodic behavior. Our results for first Lyapunov value $L_1(\lambda_0)$ presented in section 2 suggest that system (2) is structurally stable for some intervals of its parameters, and therefore soft stability loss takes place.

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ЕФЕКТ НА СИНХРОНИЗАЦИЯТА ВЪРХУ ЕДНА СИСТЕМА ОТ HOPF-LANGFORD ТИП

Светослав Г. Николов[1,2]
S.Nikolov@imbm.bas.bg

[1]Висше транспортно училище „Т. Каблешков", ул. Г. Милев № 158, 1574 София,
[2]Институт по механика-Българска Академия на Науките,
ул. Акад. Г. Бончев, бл. 4, 1113 София,
РЕПУБЛИКА БЪЛГАРИЯ

Ключови думи: *анализ, синхронизация, нелинейна динамика, система на Hopf-Langford*

Резюме: *В тази статия е изследвано качествено и числено динамичното поведение на една 3D автономна дисипативна нелинейна система от Хопф-Лангфордов тип. Показано е, че 3D нелинейната оригинална система може да бъде разделена на подсистеми от синхронизационен вид водеща-подчинена, ако се включи енергията на системата. На базата на пресмятането на първата Ляпунова величина за водещата система, правим опит да дадем обща рамка (от гледна точка на бифуркационната теория) за разбирането на структурната устойчивост и бифуркационно поведение на оригиналната система. Също така е изследван ефектът на синхронизиране върху динамиката на оригиналната система чрез числени симулации.*