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ANALYSIS OF A ROSSLER TYPE DYNAMICAL SYSTEM

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Key words: *analysis, chaos, Rossler prototype-4 system*
Abstract: *In this paper we investigate a 3D autonomous dissipative nonlinear system of ODEs- Rossler prototype-4 system. The analysis reveals that the system may exhibit the phenomena of Shilnikov chaos. Further, it is shown via analytical calculations that the considered system can be presented in the form of a linear oscillator with one nonlinear automatic regulator. Finally, it is found that for some new combinations of parameters, the system demonstrates chaotic behavior and transition from chaos to regular behavior is realized through inverse period-doubling bifurcations.*

1. INTRODUCTION

In the last thirty years, many authors have been intrigued by the quest for the mathematically simplest systems of various types that can exhibit chaos [1-5]. An good example is the book of J. Sprott [4] in which he discover: 1) some new systems that are simpler than those previously know; 2) these new systems are otherwise more "elegant" by virtue of the number of parameters- their values, special symmetry and economy of notation.

It is well-known that chaos cannot occur in dissipative two dimensional systems (one degree of freedom) of ordinary differential equations (ODEs). Chaos requires at systems with least one and a half degrees of freedom. Such systems have so-called strange attractors (the trajectory winds around forever, never repeating) with noninteger dimension. In a three-dimensional continuous dissipative dynamical systems the only possible spectra, and the attractors, are as follows: a strange attractor, a two-torus, a limit cycle, a fixed point [6-8]

Mathematical representation of a spatial order and chaos are saddle equilibria, saddle periodic movements or complex saddle invariant set. According to [9], around a saddle-focus equilibrium a systematic characterization of homoclinicity can be provided. In other words, if the Shilnikov condition is satisfied, i.e. the saddle-focus index $\delta = \left| \mathrm{Re}\left(\chi_2 / \chi_1 \right) \right| < 1$ (where χ_1 and χ_2 are the leading eigenvalues), then an infinity number of non-periodic trajectories coexist in the vicinity of a homoclinic trajectory bi-asymptotic to the saddle-focus. For more details see [10].

For a long period of time the Lorenz and Rossler systems were regarded as the simplest chaotic autonomous dissipative systems of ODEs. In this paper we consider the so-called *Rossler prototype-4 system* given by

(1)
$$\dot{x} = -y - z, \quad \dot{y} = x, \quad \dot{z} = -bz + a\left(y - y^2\right),$$

where $(x, y, z) \in R^3$ are the state variables. It is seen that system (1) has only six terms, a single quadratic nonlinearity and two parameters. The system was numerically studied in [1, 3, 4]. It was shown that for $a = b = 0.5$ the system (1) has chaotic behavior, and for $a \in (0,1]$, $b \in (0,1]$ has different dynamics- unbounded, periodic and chaotic solutions.

In this paper, using analytical and numerical tools, we investigate the dynamical behavior of system (1). The plan is as follows: in Section 2 we present analytical and numerical results concerning the system (1). In Section 3 we summarize our results.

2. ANALYTICAL AND NUMERICAL RESULTS

In this section, we consider the system (1) which present an autonomous dynamical model. The equilibrium (steady state) points of system (1) are: $O_1(0,0,0)$ and $O_2\left(0, 1+\frac{b}{a}, -1-\frac{b}{a}\right)$. The divergence of the flow (1) is:

(2)
$$D_3(t) = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = D_3(t_0) \times e^{-bt} = -b,$$

where $D_3(t_0)$ is a volume element. The system (1) is dissipative, when $D_3(t) < 0$, i.e. $b > 0$.

In order to determine the character of fixed points O_1 and O_2, we make the following substitutions into (1):

(3)
$$x = x_1 + \bar{x}, \; y = x_2 + \bar{y}, \; z = x_3 + \bar{z}.$$

Hence, after some transformations, the system (1) has the form:

(4)
$$\dot{x}_1 = -x_2 - x_3, \quad \dot{x}_2 = x_1, \quad \dot{x}_3 = cx_2 - bx_3 - ax_2^2,$$

where $c = a(1 - 2\bar{y})$. The Jacobian matrixes associated to the system (4) at O_1 and O_2 are:

(5)
$$J_{O_1} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & a & -b \end{pmatrix}, \quad J_{O_2} = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & -a-2b & -b \end{pmatrix}.$$

Thus, the characteristic equation possesses the form:

(6)
$$\chi^3 + p\chi^2 + q\chi + r = 0.$$

According to the Routh-Hurwitz criteria the fixed points of system (1) will be stable if $p = b > 0$, $q = 1 > 0$, $r^{(1)} = a + b > 0$, $r^{(2)} = -a - b > 0$, $R^{(1)} = pq - r^{(1)} = -a > 0$ and $R^{(2)} = pq - r^{(2)} = a + 2b > 0$. For $a \in [-1.5, 1]$ and $b \in (0.29, 0.8]$ we obtain that the two fixed points O_1 and O_2 at bifurcation parameters a and/or b are from *saddle-focus* type (*i)* negative real eigenvalue and complex eigenvalues with positive real part (unstable focus); *ii)* positive real eigenvalue and complex eigenvalues with negative real part (stable focus)), and for some subintervals the fixed point O_2 can be from *stable focus* type. These two fixed points can be included in homoclinic/heteroclinic structures of Shilnikov type, where their invariant manifolds W^S and W^U, are meeting each other in a most intricate manner.

For $a = -1.5, b = 0.6$ and $a = -1, b = 0.35$ the equilibriums and their eigenvalues are given by:

$O_1(0, 0, 0)$, then $(\chi_1, \chi_2, \chi_3) = (0.5508, -0.5754 \pm 1.1414i)$;
$O_2(0, 0.6, -0.6)$ then $(\chi_1, \chi_2, \chi_3) = (-0.7855, 0.0928 \pm 1.0664i)$.

$O_1(0, 0, 0)$, then $(\chi_1, \chi_2, \chi_3) = (0.4694, -0.4097 \pm 1.1031i)$;
$O_2(0, 0.6, -0.6)$ then $(\chi_1, \chi_2, \chi_3) = (-0.5754, 0.1127 \pm 1.0569i)$.

It is easy to see that the system (1) can be presented in the form of a (non)-linear oscillator with one automatic regulator. In general form we have

(7) $$\ddot{y} = -\frac{\partial V}{\partial y}, \quad \dot{z} = -\varepsilon[z - g(y)],$$

where ε is a small parameter, $g(y)$ is a nonlinear polynomial function and V is the potential in the form $V = \frac{\alpha_0 y^4}{2} - \sum_{i=1}^{2} (\alpha_i - \beta_i z) \frac{y^i}{i}$ [11]. For system (1) we have: $\alpha_0 = \alpha_1 = \beta_2 = 0$, $\alpha_2 = -1$ and $\beta_1 = 1$. Hence, for the system (1) we can write

(8) $$\ddot{y} = -\frac{\partial V}{\partial y} = -y - z, \qquad \dot{z} = -b[z - kg(y)] = -bz + a(y - y^2),$$

where $k = a/b$, and the potential of the system is $V = \frac{y^2}{2} + yz$. The energy of the system (1) is $E = V + \frac{\dot{y}^2}{2}$. Here we note that Lorenz system also can be presented in the form (7) as $\alpha_0 = 1, \alpha_1 = \alpha_2 = \beta_1 = 0$, $\beta_2 = 1$ and $g(y) = y^2$ [11].

It is well-known that a homoclinic/heteroclinic cycle is one of the common scenarios of the appearance of chaotic behavior. In our case here, the known analytical approaches in this direction are not applicable [12, 13], and we are forced to use numerical simulations of our system (1). Hence, we find new results for its bifurcation behavior and routes to chaos.

Figure 1 shows bifurcation diagrams for the system (1): (a) values of y coordinate and (b) values of z coordinate are plotted against b regarded as a continuously varying control (bifurcation) parameter, when the parameter $a = -1.5$ is fixed. It is seen that for

$b \in [0.495, 0.525]$ the system is chaotic. On a further increase of the bifurcation parameter b, the system (1) exhibits inverse period doubling bifurcations leading to a periodic motion.

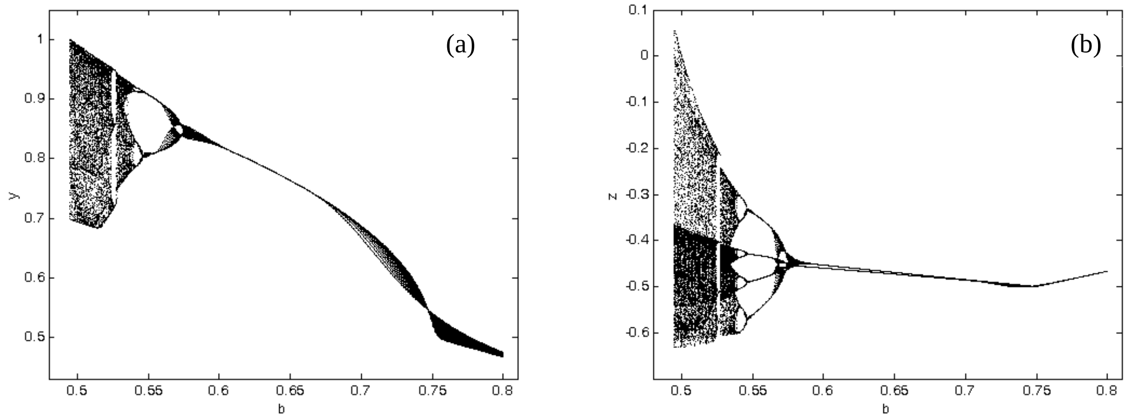

Figure.1. Bifurcation diagrams: a) $y(t)$ versus b; and b) $z(t)$ versus b, generated by computer solution of the system (1) at $a = -1.5$. The initial conditions are $x(0) = y(0) = z(0) = 0.1$. Note that $b \in [0.495, 0.8]$.

In Figure 2 the bifurcation diagrams of the system (1) are shown. It can be seen that at $b \in [0.315, 0.32]$ chaotic solution occurs. In analogy with the previous case, the system passes from chaos to regular motion after inverse period doubling bifurcations. It is interesting to note that here the system (1) has regular solutions at the beginning and in the end of the interval for the control parameter b. Comparing Fig.1 and Fig. 2 we conclude that in the case, when $a = -1.5$ the chaotic zone is longer than those obtained in Fig. 2 for $a = -1$.

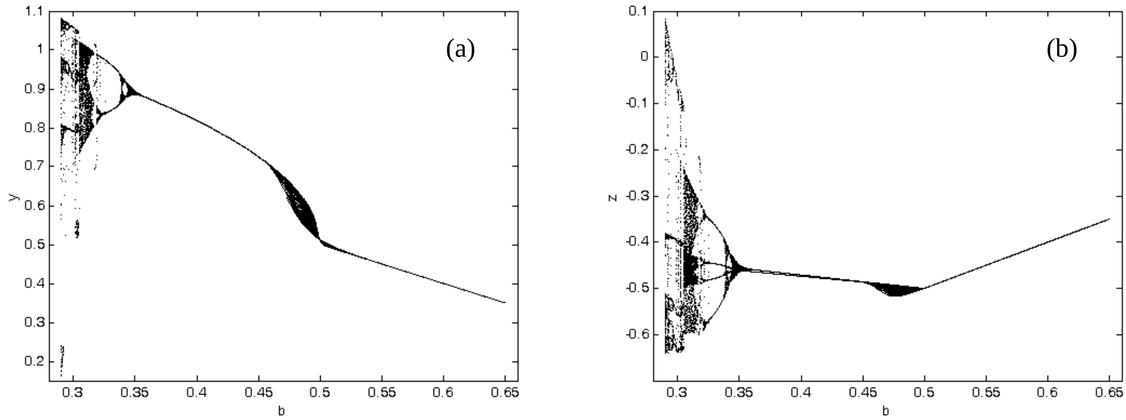

Figure 2. Bifurcation diagrams: a) $y(t)$ versus b; and b) $z(t)$ versus b, generated by computer solution of the system (1) at $a = -1$. The initial conditions are $x(0) = y(0) = z(0) = 0.1$. Note that $b \in [0.29, 0.65]$.

3. CONCLUSION

The paper presents a study of the dynamic behavior of the so-called *Rossler prototype-4 system*, using analytical and numerical tools. The governing equations were solved numerically using MATLAB (The MathWorks, Inc., Natick, MA, USA). For all simulations the initial conditions were $x(0) = y(0) = z(0) = 0.1$. We find that: 1) the system (1) has two fixed points from saddle-focus type, and therefore homoclinic/heteroclinic structures of Shilnikov type take place; 2) for values of the coefficients a and b different from these in [4], the system (1) has chaotic solutions; 3) the original system (1) can be presented in the form of a linear oscillator with one nonlinear automatic regulator.

Finally, the proposed study is a first step to the profound and fill analysis of the system (1).

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АНАЛИЗ НА ЕДНА ДИНАМИЧНА СИСТЕМА ОТ РЬОСЛЕРОВ ВИД

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Ключови думи: *анализ, хаос, прототип -4 Рьослерова система*

Резюме: *В тази статия изследваме една 3D автономна дисипативна нелинейна система от ОДУ- Rossler prototype-4 системата. Анализът ѝ показва, че тя може да прояви явлението Шилников хаос. По-нататък е показано чрез аналитични пресмятания, че изследваната система може да бъде предстевена във вид на линеен осцилатор с нелинеен автоматичен регулатор. Най-накрая е намерено, че за някои нови комбинации на стойностите на параметрите, системата показва хаотично поведение като преходът от хаос към регулярно поведение се реализира чрез обратни бифуркации на удвояване на периода.*