Mechanics Transport Communications

ISSN 1312-3823 (print)
ISSN 2367-6620 (online)
volume 17, issue 3, 2019

Academic journal http://www.mtc-aj.com article № 1840

DYNAMICS OF A HAMILTONIAN SYSTEM WITH FOUR DEGREES OF FREEDOM

Svetoslav Nikolov[1,2], Vassil Zlatanov[3]
S.Nikolov@imbm.bas.bg; vassil60@abv.bg

[1]University of Transport, Geo Milev Str., 158, 1574 Sofia,
[2]Institute of Mechanics, Bulgarian Academy of Sciences, Sofia,
[3]University of Food Technologies, Maritza Str., 26, 4002 Plovdiv,
BULGARIA

***Key words**:Hamiltonian system, nonlinearity, nonintegrability, chaos*

***Abstract:** Recently, there has been an increasing interest in high-dimensional autonomous Hamiltonian systems. In this paper, we investigate the dynamics of a Hamiltonian system with four degrees of freedom. Based on our qualitative analysis, we obtain that this system has a whole plane of unstable fixed points and therefore the occurrence of chaotic behaviour is possible.*

1. INTRODUCTION

The investigation of dynamics of autonomous Hamiltonian systems with more than three degrees of freedom (so-called high dimensional systems) is an important problem in modern nonlinear mechanics [1, 3]. In the last two decades, this problem has been studied by many authors, but a general theory for systems with dimension higher than three is still missing [2, 4]. The complexity in Hamiltonian systems is due to transition from local to global behaviour, as with the increase in dimension many new phenomena arise [3].

In this paper we consider a mechanical system (oscillator) with four degrees of freedom, depicted in Figure 1. For this system the kinetic energy T and the potential energy U have the form

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{m_3}{2}\left(\dot{x}_1^2 + l^2\dot{\phi}_1^2 - 2l\dot{x}_1\dot{\phi}_1\cos\phi_1\right) + \frac{m_4}{2}\left(\dot{x}_2^2 + l^2\dot{\phi}_2^2 - 2l\dot{x}_2\dot{\phi}_2\cos\phi_2\right), \tag{1}$$

$$U = \frac{1}{2}c_1x_1^2 + \frac{1}{2}c_2\left(x_2 - x_1\right)^2 + \frac{1}{2}c_3\left(x_2 - x_1 + l\sin\phi_1 - l\sin\phi_2\right)^2 + m_3gl\left(1-\cos\phi_1\right) + m_4gl\left(1-\cos\phi_2\right),$$

where $m_1 \div m_4$ are the masses, l is the length of the massless rods, c_1, c_2 and c_3 are the spring (stiffness) constants, g is the acceleration of gravity, x_1, x_2, ϕ_1 and ϕ_2 are the generalized coordinates. Here we note that the velocities v_3^2 and v_4^2 of the mass-points m_3 and m_4 are the sum of the projections v_x and v_y, i.e.

(2) $$v_3^2 = \left(\dot{x}_1 - l\dot{\phi}_1\cos\phi_1\right)^2 + l^2\dot{\phi}_1^2\sin^2\phi_1 = \dot{x}_1^2 + l^2\dot{\phi}_1^2 - 2l\dot{x}_1\dot{\phi}_1\cos\phi_1,$$
$$v_4^2 = \left(\dot{x}_2 - l\dot{\phi}_2\cos\phi_2\right)^2 + l^2\dot{\phi}_2\sin^2\phi_2 = \dot{x}_2^2 + l^2\dot{\phi}_2^2 - 2l\dot{x}_2\dot{\phi}_2\cos\phi_2.$$

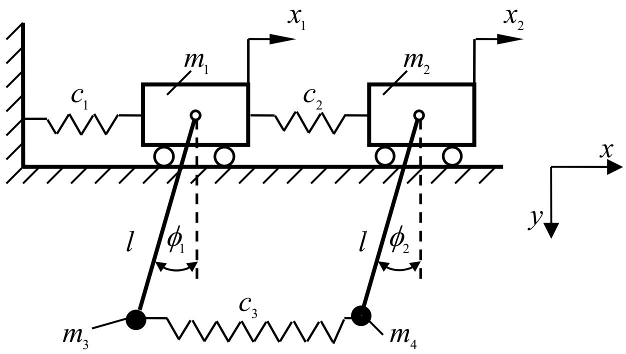

Figure 1. Schematic diagram of a system with four degrees of freedom.

Here we are interested in the dynamical behaviour of the mechanical system shown in Figure 1, and especially in the type of its fixed points.

2. HAMILTONIAN FORM

In Hamiltonian formulation, the state of a system is characterized not only by its positions (generalized coordinates) $q_i\ (i = 1,...,n)$ but also by its momenta p_i. Hence, the Hamiltonian of the mechanical system depicted in Figure 1 is

(3) $$H = T + U = \frac{1}{2\Delta_1}p_1^2 + \frac{1}{2\Delta_2}p_2^2 + \frac{m_1+m_3}{2m_3l^2\Delta_1}p_3^2 + \frac{m_2+m_4}{2m_4l^2\Delta_2}p_4^2 + $$
$$+\frac{1}{l}\left(\frac{\cos\phi_1}{\Delta_1}p_1p_3 + \frac{\cos\phi_2}{\Delta_2}p_2p_4\right) + \frac{1}{2}c_1x_1^2 + \frac{1}{2}c_2\left(x_2 - x_1\right)^2 +$$
$$+\frac{1}{2}c_3\left(x_2 - x_1 + l\sin\phi_1 - l\sin\phi_2\right)^2 + m_3gl\left(1-\cos\phi_1\right) + m_4gl\left(1-\cos\phi_2\right).$$

where $p_1 = \partial T/\partial\dot{x}_1$, $p_2 = \partial T/\partial\dot{x}_2$, $p_3 = \partial T/\partial\dot{\phi}_1$, $p_4 = \partial T/\partial\dot{\phi}_2$ are the generalized momenta, and $\Delta_1 = \left(m_1 + m_3 - m_3\cos^2\phi_1\right)$, $\Delta_2 = \left(m_2 + m_4 - m_4\cos^2\phi_2\right)$.

Using the Hamiltonian formalism, we obtain the canonical equations of motion

$$\dot{x}_1 = \frac{\partial H}{\partial p_1} = \frac{1}{\Delta_1}\left(p_1 + \frac{\cos\phi_1}{l}p_3\right),\quad \dot{p}_1 = -\frac{\partial H}{\partial x_1} = -a_1x_1 + a_2x_2 + c_3l\left(\sin\phi_1 - \sin\phi_2\right),$$
$$\dot{x}_2 = \frac{\partial H}{\partial p_2} = \frac{1}{\Delta_2}\left(p_2 + \frac{\cos\phi_2}{l}p_4\right),\quad \dot{p}_2 = -\frac{\partial H}{\partial x_2} = a_2\left(x_1 - x_2\right) - c_3l\left(\sin\phi_1 - \sin\phi_2\right),$$
(4) $$\dot{\phi}_1 = \frac{\partial H}{\partial p_3} = \frac{1}{l\Delta_1}\left(p_1\cos\phi_1 + \frac{m_1+m_3}{m_3l}p_3\right),\quad \dot{p}_3 = -\frac{\partial H}{\partial\phi_1} = \frac{\sin\phi_1}{l^2\Delta_1^2}\left[m_3l^2p_1^2\cos\phi_1 + l\Delta_3p_1p_3 +\right.$$
$$\left.+\left(m_1+m_3\right)p_3^2\cos\phi_1\right] + c_3l\left(x_1 - x_2\right)\cos\phi_1 - c_3l^2\left(\sin\phi_1 - \sin\phi_2\right)\cos\phi_1 - m_3gl\sin\phi_1,$$
$$\dot{\phi}_2 = \frac{\partial H}{\partial p_4} = \frac{1}{l\Delta_2}\left(p_2\cos\phi_2 + \frac{m_2+m_4}{m_4l}p_4\right),\quad \dot{p}_4 = -\frac{\partial H}{\partial\phi_2} = \frac{\sin\phi_2}{l^2\Delta_2^2}\left[m_4l^2p_2^2\cos\phi_2 + l\Delta_4p_2p_4 +\right.$$
$$\left.+\left(m_2+m_4\right)p_4^2\cos\phi_2\right] - c_3l\left(x_1 - x_2\right)\cos\phi_2 + c_3l^2\left(\sin\phi_1 - \sin\phi_2\right)\cos\phi_2 - m_4gl\sin\phi_2,$$

where $a_1 = c_1 + c_2 + c_3$, $a_2 = c_2 + c_3$, $\Delta_3 = m_1 + m_3 + m_3 \cos^2 \phi_1$ and $\Delta_4 = m_2 + m_4 + m_4 \cos^2 \phi_2$.

In fact, since the Hamiltonian in (3) does not depend explicitly on time t, the Hamiltonian function H defines a first integral (called also the generalized integral of energy). Moreover, according to Liouville-Arnold theorem [5], it can be shown that the knowledge of $n = 4$ degrees of freedom, first integrals f_l $(l = 1,...,4)$ which are functionally independent, allows us to trivially integrate the equations of the motion. Note that the motion region is bounded if $l = 4$ and unbounded if $l \neq 4$. More generally, if $\dim M_f = 4$ (where level manifold M_f is compact and connected), the compact hypersurface corresponding to $l = n = 4$ is called a four-torus T^4. It is important to remark that the system (4) is with four degrees of freedom and belongs to the class of nonintegrable Hamiltonian systems, i.e. the foliation in invariant tori is completely destroyed. The eight-dimensional phase space $Z = \{(x_1, x_2, \phi_1, \phi_2, p_1, p_2, p_3, p_4)\}$ is characterized by an extremely complex behaviour of the phase trajectories and by existence of instability zones in which the motion has chaotic (nonregular) character [5, 6, 7].

By taking

$$\frac{\partial H}{\partial p_1} = \frac{\partial H}{\partial p_2} = \frac{\partial H}{\partial p_3} = \frac{\partial H}{\partial p_4} = \frac{\partial H}{\partial x_1} = \frac{\partial H}{\partial x_2} = \frac{\partial H}{\partial \phi_1} = \frac{\partial H}{\partial \phi_2} = 0, \tag{5}$$

we obtain the fixed points of the system (4), i.e.

$$\begin{aligned} &\bar{p}_1 = \bar{p}_2 = \bar{p}_3 = \bar{p}_4 = \bar{x}_1 = 0,\ \bar{x}_2 = -\frac{c_3 l}{c_2 + c_3}\left(\sin\bar{\phi}_1 - \sin\bar{\phi}_2\right), \\ &\bar{\phi}_1 = \arctan\left(-\frac{m_4}{m_3}\tan\bar{\phi}_2\right),\ \bar{\phi}_2 \neq \pm\frac{(2k+1)\pi}{2}\ (k = 0,1,2,...). \end{aligned} \tag{6}$$

Here we note that the function tg is odd one and $tg\bar{\phi}_2 \to \pm\infty$ when $\bar{\phi}_2 = \pm\frac{(2k+1)\pi}{2}$. Thus, we assume that $\bar{\phi}_2$ (a relative (non-simple) equilibrium) always lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and between $\frac{\pi}{2}\left(-\frac{\pi}{2}\right)$ and $\frac{3\pi}{2}\left(-\frac{3\pi}{2}\right)$, i.e.

$$\bar{\phi}_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ and } \bar{\phi}_2 \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right). \tag{7}$$

According to [8], if the corresponding linearized system (of a nonlinear system) is non-simple then the fixed point(s) is (are) said to be also non-simple. In this case such linear systems contain a line, or possibly a whole plane, of fixed points. The nature of the local phase portrait is determined by nonlinear terms and there are infinitely many different types of local phase portraits.

In Figure 2, lines of fixed points (non-simple fixed points) in $(\bar{\phi}_1, \bar{\phi}_2)$ plane are shown, when into (6) $\bar{\phi}_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\bar{\phi}_2 \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and $\frac{m_4}{m_3} \in [0.1, 2]$.

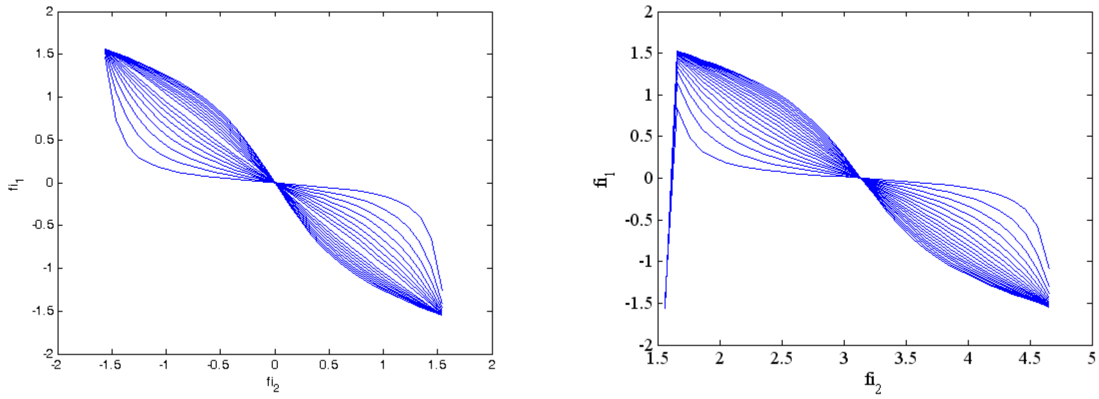

Figure 2. Lines of fixed points for system (4) in $(\bar{\phi}_1, \bar{\phi}_2)$ plane, when $\bar{\phi}_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ - left panel and $\bar{\phi}_2 \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ - right panel, and $\frac{m_4}{m_3} \in [0.1, 2]$.

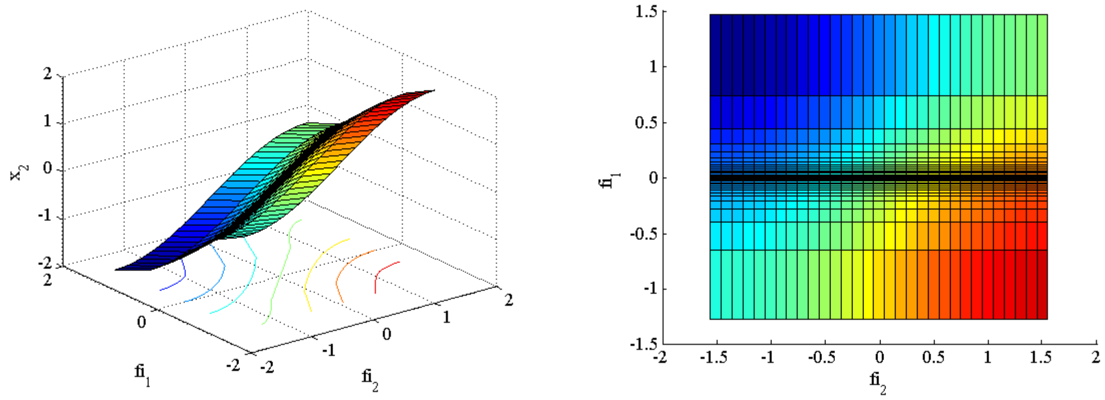

Figure 3. Surface of coordinate $\bar{x}_2$ (of fixed points for system (4)), when $\frac{c_3 l}{c_2 + c_3} = 1[m]$, and $m_4 / m_3 = 0.1$.

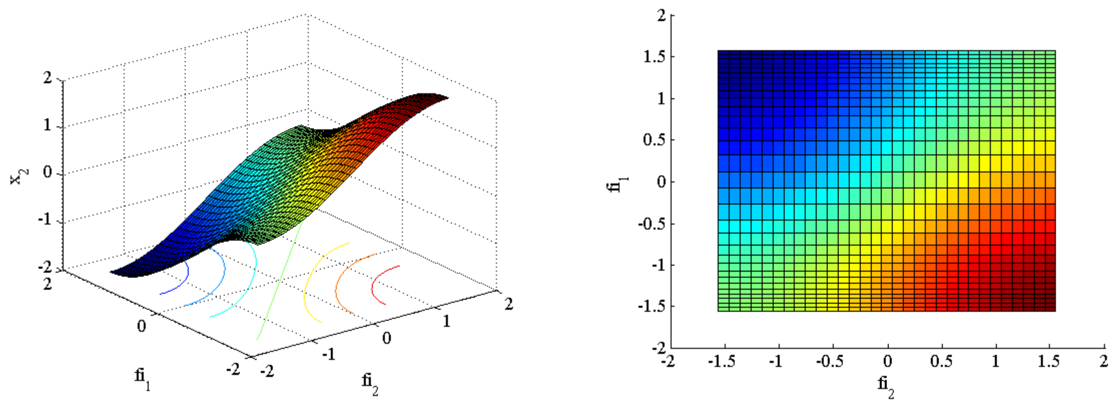

Figure 4. Surface of coordinate $\bar{x}_2$ (of fixed points for system (4)), when $\frac{c_3 l}{c_2 + c_3} = 1[m]$, and $m_4 / m_3 = 2$.

Now, we investigate the linear (local) stability of fixed points (6) of Hamiltonian system (4). According to this analysis, we first need to linearize the Hamiltonian system (4), with regards to its fixed points. The resulting set of linear ODEs is written in terms of the 8×8 Jacobian matrix $J = \{J_{jk}\}, (j, k = 1, 2, ..., 8)$. Here, the elements of this matrix are:

(8)

$$
\begin{aligned}
& J_{11}=J_{12}=J_{14}=J_{16}=J_{18}=J_{21}=J_{22}=J_{23}=J_{25}=J_{27}= \\
& =J_{31}=J_{32}=J_{34}=J_{36}=J_{38}=J_{41}=J_{42}=J_{43}=J_{45}=J_{47}= \\
& =J_{55}=J_{56}=J_{57}=J_{58}=J_{65}=J_{66}=J_{67}=J_{68}=J_{76}=J_{78}= \\
& =J_{85}=J_{87}=0,\ J_{13}=-J_{75}=\frac{\partial^2 H}{\partial \phi_1 \partial p_1}=-\frac{\sin\bar{\phi}_1}{\bar{\Delta}_1^2}\left(2m_3\bar{p}_1\cos\bar{\phi}_1+\frac{\bar{\Delta}_3}{l}\bar{p}_3\right), \\
& J_{15}=\frac{\partial^2 H}{\partial p_1^2}=\frac{1}{\Delta_1},\ J_{17}=J_{35}=\frac{\partial^2 H}{\partial p_1 \partial p_3}=\frac{\cos\bar{\phi}_1}{l\bar{\Delta}_1},\ J_{24}=-J_{86}=\frac{\partial^2 H}{\partial \phi_2 \partial p_2}= \\
& =-\frac{\sin\bar{\phi}_2}{\bar{\Delta}_2^2}\left(2m_4\bar{p}_2\cos\bar{\phi}_2+\frac{\bar{\Delta}_4}{l}\bar{p}_4\right),\ J_{26}=\frac{\partial^2 H}{\partial p_2^2}=\frac{1}{\bar{\Delta}_2},\ J_{28}=J_{46}=\frac{\partial^2 H}{\partial p_2 \partial p_4}= \\
& =\frac{\cos\bar{\phi}_2}{l\bar{\Delta}_2},\ J_{33}=-J_{77}=\frac{\partial^2 H}{\partial \phi_1 \partial p_3}=-\frac{\sin\bar{\phi}_1}{l\bar{\Delta}_1^2}\left[\bar{\Delta}_3\bar{p}_1+\frac{2(m_1+m_3)}{l}\bar{p}_3\cos\bar{\phi}_1\right], \\
& J_{37}=\frac{\partial^2 H}{\partial p_3^2}=\frac{m_1+m_3}{m_3 l^2\bar{\Delta}_1},\ J_{44}=-J_{88}=-\frac{\sin\bar{\phi}_2}{l\bar{\Delta}_2^2}\left[\bar{\Delta}_4\bar{p}_2+\frac{2(m_2+m_4)}{l}\bar{p}_4\cos\bar{\phi}_2\right], \\
& J_{48}=\frac{\partial^2 H}{\partial p_4^2}=\frac{m_2+m_4}{m_4 l^2\bar{\Delta}_2},\ J_{51}=-\frac{\partial^2 H}{\partial x_1^2}=-(c_1+c_2+c_3),\ J_{52}=J_{61}=-\frac{\partial^2 H}{\partial x_1 \partial x_2}=c_2+c_3, \\
& J_{53}=J_{71}=-\frac{\partial^2 H}{\partial x_1 \partial \phi_1}=lc_3\cos\bar{\phi}_1,\ J_{54}=J_{81}=-\frac{\partial^2 H}{\partial x_1 \partial \phi_2}=-lc_3\cos\bar{\phi}_2, \\
& J_{62}=-\frac{\partial^2 H}{\partial x_2^2}=-(c_2+c_3),\ J_{63}=J_{72}=-\frac{\partial^2 H}{\partial x_2 \partial \phi_1}=-lc_3\cos\bar{\phi}_1,\ J_{64}=J_{82}= \\
& =-\frac{\partial^2 H}{\partial x_2 \partial \phi_2}=lc_3\cos\bar{\phi}_2,\ J_{73}=-\frac{\partial^2 H}{\partial \phi_1^2}=\frac{1}{\bar{\Delta}_1^3}\left\{\frac{\cos\bar{\phi}_1}{l}\bar{p}_1\bar{p}_3\left\{\left[\bar{\Delta}_1+2m_3\left(1-2\sin^2\bar{\phi}_1\right)\right]\bar{\Delta}_1-\right.\right. \\
& \left.\left.-4m_3\Delta_3\sin^2\bar{\phi}_1\right\}+\left[\left(1-2\sin^2\bar{\phi}_1\right)\bar{\Delta}_1-4m_3\sin^2\bar{\phi}_1\cos^2\bar{\phi}_1\right]\left(m_3p_1^2+\frac{m_1+m_3}{l^2}p_3^2\right)\right\}- \\
& -c_3l\left[l\left(1-2\sin^2\bar{\phi}_1+\sin\bar{\phi}_1\sin\bar{\phi}_2\right)-\left(\bar{x}_2-\bar{x}_1\right)\sin\bar{\phi}_1\right]-m_3gl\cos\bar{\phi}_1, \\
& J_{74}=J_{83}=-\frac{\partial^2 H}{\partial\bar{\phi}_1\partial\bar{\phi}_2}=c_3l^2\cos\bar{\phi}_1\cos\bar{\phi}_2,\quad J_{84}=\frac{1}{\bar{\Delta}_2^3}\left\{\frac{\cos\bar{\phi}_2}{l}\bar{p}_2\bar{p}_4\times\right. \\
& \times\left\{\left[\bar{\Delta}_2+2m_4\left(1-2\sin^2\bar{\phi}_2\right)\right]\bar{\Delta}_2-4m_4\bar{\Delta}_4\sin^2\bar{\phi}_2\right\}+\left[\left(1-\sin^2\bar{\phi}_2\right)\bar{\Delta}_2-4m_4\sin^2\bar{\phi}_2\cos^2\bar{\phi}_2\right]\times \\
& \left.\times\left(m_4\bar{p}_2^2+\frac{m_2+m_4}{l^2}\bar{p}_4^2\right)\right\}-c_3l\left[l\left(1-2\sin^2\bar{\phi}_2+\sin\bar{\phi}_1\sin\bar{\phi}_2\right)+\left(\bar{x}_2-\bar{x}_1\right)\sin\bar{\phi}_2\right]-m_4gl\cos\bar{\phi}_2.
\end{aligned}
$$

The 8×8 Jacobian matrix at (6) has eigenvalues

(9)
$$\lambda\left(\lambda^7-\beta_2\lambda^5-\beta_1\lambda^3-1\right)=0,$$

where

(10)
$$
\begin{aligned}
& \beta_1=J_{35}\left(J_{73}J_{17}J_{51}+J_{63}J_{26}J_{52}\right)+J_{15}J_{54}\left(J_{48}J_{81}+J_{46}J_{61}\right), \\
& \beta_2=J_{26}J_{62}+J_{28}J_{82}+J_{15}J_{51}+J_{17}J_{71}+J_{35}J_{53}+J_{46}J_{64}+J_{37}J_{73}+J_{48}J_{84}.
\end{aligned}
$$

It is seen that (9) has a zero root. Thus, according to [9], we conclude that depending on the number of positive (negative) roots, we can face one of the following cases or combination between them: a compound saddle-focus, a compound-knot etc. Therefore, the system (4) has homo(hetero)clinic structure and transition to chaos (occurrence of chaotic behaviour) takes place [9, 10, 11].

4. CONCLUDING REMARKS

The present paper studies the dynamics of a Hamiltonian system with four degrees of freedom. Specially, we focused our estimations on the local properties of fixed points. Hence, a whole plane of unstable fixed points (from type compound saddle-focus or compound saddle-knot) was obtained. Thus, according to the theory of dynamical systems, the model (4) has homoclinic/heteroclinic structure and occurrence of chaotic behaviour is possible. Finally, we note that the proposed investigation is an initial step to profound and full analysis of the system (4).

REFERENCES

[1] Arnold, V., Kozlov, V., Neishtadt, A., Mathematical aspects of classical and celestial mechanics, 3rd. edition, Springer, NY, 2006.

[2] Skokos, Ch., On the stability of periodic orbits of high dimensional autonomous Hamiltonian systems, Physica D, vol. 159, pp. 155-179, 2001.

[3] Bountis, T., Skokos, H., Complex Hamiltonian dynamics, Springer, NY, 2012.

[4] Nikolov, S., Zaharieva, D., Dynamical behaviour of compound elastic pendulum, MATEC Web of Conferences, vol. 145, art. No 01003, 10 pages, 2018.

[5] Arnold, V., Mathematical methods of classical mechanics. Springer-Verlag, NY, 1978.

[6] Vilasi, G., Hamiltonian dynamics. World Scientific, Singapore, 2001.

[7] Wiggins, S., Global bifurcations and chaos. Analytical methods. Springer-Verlag, NY, 1988.

[8] Arrosmith, D., Place, C., Dynamical systems. Differential equations, maps and chaotic behaviour. Chapman & Hall, London, 1992.

[9] Nikolov, S., Estimating of bifurcations and chaotic behaviour in a four-dimensional system, J. of the Calcutta Math. Society, vol. 2, No 1, pp. 17-28, 2006.

[10] Ovsyannikov, I., Shilnikov, L., On systems with a saddle-focus homoclinic curve, Math. Sbornik, vol. 130(172), No 4(8), pp. 552-570, 1986.

[11] Harterich, J., Cascades of reversible homoclinic orbits to a saddle-focus equilibrium, Physica D: Nonlinear Phenomena, vol. 112, No 1-2, pp. 187-200, 1998.

ДИНАМИКА НА ЕДНА ХАМИЛТОНОВА СИСТЕМА С ЧЕТИРИ СТЕПЕНИ НА СВОБОДА

Светослав Николов[1,2], Васил Златанов[3]

S.Nikolov@imbm.bas.bg; vassil60@abv.bg

[1]Висше транспортно училище „Т. Каблешков", ул. Г. Милев, № 158, 1574 София,
[2]Институт по механика-БАН, ул. Акад. Г. Бончев., бл. 4, 1113 София,
[3]Университет по хранителни технологии, ул. Марица, № 26, 4002 Пловдив, БЪЛГАРИЯ

Ключови думи: *хамилтонова система, нелинейност, неинтегруемост, хаос*

Резюме: *През последните години се наблюдава засилен интерес към високо размерни автономни хамилтонови системи. В тади статия, ние изследваме динамиката на една хамилтонова система с четири степени на свобода. На основата на направения качествен анализ ние получаваме, че тази система има една цяла равнина от неусточиви фиксирани точки и следователно е възможна появата на хаотично поведение.*