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BIFURCATION BEHAVIOR OF A HAMILTONIAN SYSTEM WITH TWO DEGREES OF FREEDOM

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Abstract: *The investigation of the qualitative changes in the dynamics of a system is object of bifurcation theory. In this paper we analyze the bifurcation behavior of a Hamiltonian system with two degrees of freedom describing the rider and the swing (pumped from the seated position) as a compound pendulum. Our analytical calculations predict that a Hamiltonian Hopf bifurcation (1:-1 resonance) takes place.*

1. INTRODUCTION

The dynamical system defined by canonical equations

(1) $$\dot{q} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q},$$

where $H(q,p,t) = H(x,t)$ is a Hamiltonian function of an even number of variables $q = (q_1, q_2, ..., q_l)$ and $p = (p_1, p_2, ..., p_l)$, and l is the number of freedoms, is called *Hamiltonian system*. If the Hamiltonian is independent of time t, the system is autonomous as $H(q,p)$ is a constant or also integral of motion.

There is a standard procedure for determining the stability of equilibria of an ordinary differential equation (ODE)

(2) $$\dot{y} = f(y)$$

where $y = (y_1, ..., y_n)$ and f is a smooth function. Equilibria points $\bar{y}$ are such that $f(\bar{y}) = 0$, i.e. these points are fixed in time under the dynamics. By stability of the fixed point $\bar{y}$ we mean that any solution to $\dot{y} = f(y)$ that starts near $\bar{y}$ remains close to $\bar{y}$ throughout all time. A typical method of determining the stability of $\bar{y}$ is to calculate the first variation equation

(3) $$\dot{\xi} = Df(\bar{y})\xi,$$

where $Df(\bar{y})$ is the Jacobian of f at $\bar{y}$, defined to be the matrix of partial derivatives

(4) $$Df(\bar{y}) = \left(\frac{\partial f_i}{\partial y_j} \right)_{y=\bar{y}} .$$

In this case the following theorem is valid [1]:
Lyapunov Theorem. *If all the eigenvalues of $Df(\bar{y})$ lie in the strict left half plane, then the fixed point $\bar{y}$ is stable. If any of the eigenvalues lie in the right half plane, then the fixed point is unstable.*

It is well-known, however, that for Hamiltonian systems, the eigenvalues come in quartets that are symmetric about the origin, and so they cannot all lie in the strict left half plane [2]. Thus, the above Lyapunovs theorem for stability is not appropriate to ascertain whether or not a fixed point of a Hamiltonian system is stable. When the Hamiltonian is in canonical form, one can use Lagrange and Dirichlet method to test the stability of fixed points. Obviously, this method starts with the observation that for a fixed point $(\bar{q}, \bar{p})$ of such a system

(5) $$\frac{\partial H}{\partial q}(\bar{q}, \bar{p}) = \frac{\partial H}{\partial p}(\bar{q}, \bar{p}) = 0.$$

Hence, the fixed point occurs at a critical point of the Hamiltonian. In the case that a Hamiltonian is written in the form of kinetic T plus potential U energies, critical points occur when $\bar{p} = 0$ and $\bar{q}$ is a critical point of U. – Hence, the Lagrange-Dirichlet criterion for searching for a non-degenerate minimum of U.

The numbers of degrees of freedom (or the dimension of the configuration space) for a Hamiltonian system, is a first measure for its complexity [3]. The number of degrees of freedom is defined as one half of the maximal rank of the Poisson structure (space) *P*, the phase portraits in *P* with one degree of freedom can be obtained by intersecting the level sets of the energy with the symplectic leaves. If the rank of the Poisson space for a point drops from two to zero, then this point is an equilibrium for every Hamiltonian system on *P*. Equilibria on regular symplectic leaves are called regular equilibria.

There exist four types of linearizations of regular equilibria for systems with one degree of freedom: *hyperbolic* – the equilibrium is saddle; ii) *elliptic* – the nearby motion is periodic; iii) *vanishing* – the linearization (from co-dimension three) contains no information; iv) *parabolic* – the character of the flow is determined from the higher than two order terms of the Hamiltonian near the equilibrium, as the system depends on external parameters or is defined on a family of symplectic leaves.

It is well-known that in one degree of freedom Hamiltonian system, the regular equilibria are given by the planar singularities (or more critical points) of the Hamiltonian function. Local bifurcations (where these singularities are not stable) occur, and the whole bifurcation scenario is included in the universal unfolding of the unstable singularity. Note here, that on a symplectic surface generic Hamiltonian systems have only elliptic and hyperbolic equilibria.

In two degrees of freedom Hamiltonian systems, there are the following types of linearizations of regular equilibria: i) *hyperbolic (saddle-saddle)* – the linearization has no eigenvalues on the imaginary axis; ii) *hypo-elliptic (saddle-centre)* – the linear vector field has one pair of real eigenvalues and one pair of imaginary eigenvalues; iii) *vanishing (saddle-focus)* – all eigenvalues are complex with non-zero real part; iv) *elliptic (centre-centre)* – for a rational frequency ratio the motion is periodic, but for an irrational one the trajectories are span around invariant tori. Instabilities in generic parametrized families of two degrees of freedom Hamiltonian systems can take place through the appearance of a pair of zero roots which transfer a centre-centre to a saddle-centre or by coincidence of pure imaginary eigenvalues which transfers a centre-centre to a saddle-focus.

In contrast to one degree of freedom Hamiltonian systems, in these with two degrees of freedom a new phenomenon can be seen – one may have two pairs of purely imaginary eigenvalues in resonance. Note, that the most important is the *1:-1* resonance, which in generic 1-parameter families triggers a Hamiltonian Hopf bifurcation. This bifurcation of regular equilibria – an elliptic equilibrium becomes hyperbolic of focus-focus type, and is related to bifurcating equilibria at singular points in one degree of freedom.

In [4], Case and Swanson modeled the rider and the swing as a compound pendulum, with a massive bob, m_1 at the rider's position on the seat and the rest of the body by two other bobs, m_2 that account for the extension of the body due to the rest of the body parts-arms, legs, head and so on. For simplicity, the bobs are represented as dumbbell. In this paper we investigate the bifurcation behavior of a model of a swing pumped from the seated position, when the dumbbell is allowed to be asymmetric – see Figure 1. This model is a Hamiltonian system with two degrees of freedom and, as can be seen later, has very complicated dynamical behavior.

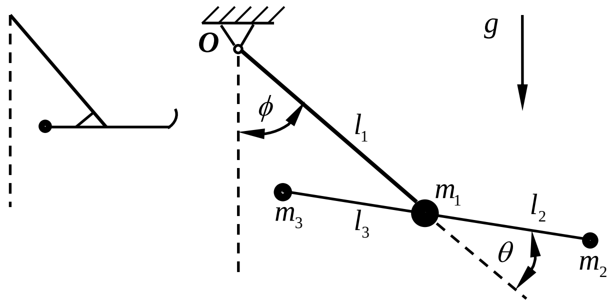

Figure 1. Person on a swing by point masses when the swinger is idealized as a rigid dumbbell with length $l_2 + l_3$, as all mass is located in point-masses $m_1 \neq m_2 \neq m_3$.

According to [5], the Hamiltonian $H(\phi, p_1, \theta, p_2)$ of the model from Fig. 1 has the form

(6)
$$H = \frac{1}{2\Delta}\left(I_2 p_1^2 + a_1 p_2^2\right) - \frac{b_1}{\Delta} p_1 p_2 - M l_1 g \cos\phi + Ng\cos(\phi+\theta),$$

where

(7)
$$a_1 = I_1 + I_2 - 2l_1 N\cos\theta,\ b_1 = I_2 - l_1 N\cos\theta,\ \Delta = \left(a_1 I_2 - b_1^2\right),$$
$$M = m_1 + m_2 + m_3,\ N = m_3 l_3 - m_2 l_2,\ I_1 = M l_1^2,\ I_2 = 2m_2 l_2^2\ .$$

Hence, the Hamiltonian system becomes

(8)
$$\dot\phi = \frac{\partial H}{\partial p_1} = \frac{1}{\Delta}\left(I_2 p_1 - b_1 p_2\right),\quad \dot\theta = \frac{\partial H}{\partial p_2} = \frac{1}{\Delta}\left(-b_1 p_1 + a_1 p_2\right),$$
$$\dot p_1 = -\frac{\partial H}{\partial \phi} = -M l_1 g\sin\phi + Ng\sin(\phi+\theta),$$
$$\dot p_2 = -\frac{\partial H}{\partial \theta} = Ng\sin(\phi+\theta) + \frac{l_1 N\cos\theta}{\Delta^2}\left[l_1 N\cos\theta\left(I_2 p_1^2 + a_1 p_2^2 - 2b_1 p_1 p_2\right) - \Delta\left(p_2^2 - p_1 p_2\right)\right].$$

This paper is organized as follows. In Section 2, by qualitative analysis, we investigate the dynamics of our equations of motion (8) near fixed point $\left(\bar{\phi},\bar{\theta},\bar{p}_1,\bar{p}_2\right)=(0,0,0,0)$. In Section 3 we summarize our results.

2. QUALITATIVE ANALYSIS

Actually, the $\left(\phi,\theta,p_1,p_2\right)$- components of (8) have the following fixed points in phase space

$$\bar{\phi}=\bar{\theta}=k\pi,\ \bar{p}_1=\bar{p}_2=0 \qquad (k=0,\pm1,...) \tag{9}$$

Transforming to new (local) phase space coordinates

$$x_1=\phi-\bar{\phi},\ x_2=\theta-\bar{\theta},\ p_1=y_1,\ p_2=y_2, \tag{10}$$

which denote deviations from the fixed points, we get

$$\bar{H}=\frac{1}{2\bar{\Delta}}\left(I_2y_1^2+\bar{a}_1y_2^2\right)-\frac{\bar{b}_1}{\bar{\Delta}}y_1y_2-Ml_1g\cos\left(k\pi+x_1\right)+Ng\cos\left(2k\pi+x_1+x_2\right). \tag{11}$$

Here $\bar{\Delta}=\bar{a}_1I_2-\bar{b}_1^2$, $\bar{a}_1=I_1+I_2-2l_1N\cos\left(k\pi+x_2\right)$ and $\bar{b}_1=I_2-l_1N\cos\left(k\pi+x_2\right)$.

Without loss of generality, henceforth we assume that $k=0$. Thus, the Hamiltonian into (11) becomes

$$\bar{H}=\frac{1}{2\bar{\Delta}}\left(I_2y_1^2+\bar{a}_1y_2^2\right)-\frac{\bar{b}_1}{\bar{\Delta}}y_1y_2-Ml_1g\cos\left(x_1\right)+Ng\cos\left(x_1+x_2\right). \tag{12}$$

The function $\frac{1}{\bar{\Delta}}$ can be written as a MacLaurin series:

$$\frac{1}{\rho+\sigma}=\frac{1}{\rho\left(1+\frac{\sigma}{\rho}\right)}=\frac{1}{\rho}\left[1-\frac{\sigma}{\rho}+\left(\frac{\sigma}{\rho}\right)^2-\left(\frac{\sigma}{\rho}\right)^3+...\right], \tag{13}$$

where $\rho=I_1I_2$ and $\sigma=-l_1^2N^2\cos^2x_2$. If we take only linear terms from (13) and after substitution of (9) into differential equations (8) (when x_2 is small (i.e. $x_2\approx1^0-6^0$)), we have

$$\begin{aligned}\dot{x}_1&=k_1y_1-k_2y_2 \quad , \quad \dot{y}_1=k_4x_1+k_5x_2+...h.o.t.,\\ \dot{x}_2&=-k_2y_1+k_3y_2 \quad , \quad \dot{y}_2=k_5x_1+k_5x_2+...h.o.t.,\end{aligned} \tag{14}$$

where

$$\begin{aligned}k_1&=\frac{1}{I_1}\left(1+\frac{l_1^2N^2}{I_1I_2}\right),\ k_2=\frac{I_2-l_1N}{I_1I_2}\left(1+\frac{l_1^2N^2}{I_1I_2}\right),\\ k_3&=\frac{I_1+I_2-2l_1N}{I_1I_2}\left(1+\frac{l_1^2N^2}{I_1I_2}\right),\ k_4=g\left(N-Ml_1\right),\ k_5=Ng.\end{aligned} \tag{15}$$

Here we note that for angles which are smaller than $0.7-0.9$ *rad*, an error in the determination of the original system solution is not more than $1-3$ % for these values of the angles.

At the linear part of Hamiltonian equations (14) we have the form $\dot{z} = Az$, where $z \equiv col(x_1, y_1, x_2, y_2)$, and the matrix A has eigenvalues (χ) given by

$$\chi = \pm\sqrt{\frac{1}{2}\left(-q \pm \sqrt{q^2 - 4s}\right)}, \tag{16}$$

where $q = \frac{cg}{I_1}\left[N\left(2 - \frac{I_1}{I_2}\right) - Ml_1\right]$, $s = -\frac{g^2 NMl_1c^2}{I_1^2 I_2^2}\left(I_1 I_2 - l_1^2 N^2\right)$ and $c = 1 + \frac{l_1^2 N^2}{I_1 I_2}$. The four eigenvalues move in the complex plane when q and s change, as shown in Figure 2. Thus, the equilibrium (9) (for $k = 0$) is a hyperbolic saddle if $q < 0, {q^2}/{4} < s$; an elliptic center if $q > 0, {q^2}/{4} = s$; and a center is $q > 0, {q^2}/{4} > s$. These are typical dispositions in the Hamiltonian Hopf bifurcation as the answer to the question: what happens beyond the instability depends on the non-linear terms higher than degree three in (14) [6, 7].

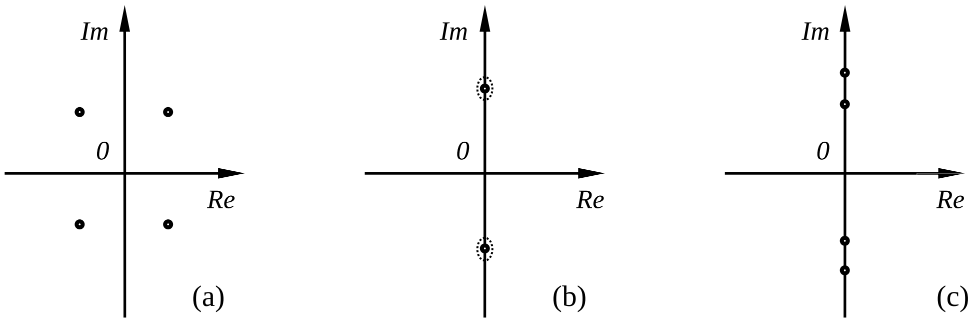

Figure 2. Disposition of eigenvalues (Eq.(16)) in the complex plane for (a) $q < 0, {q^2}/{4} < s$, (b) $q > 0, {q^2}/{4} = s$, and (c) $q > 0, {q^2}/{4} > s$.

3. CONCLUSION

In this paper we present a study of the dynamical features of a Hamiltonian system with two degrees of freedom describing the rider and the swing, pumped from the seated position (when the dumbbell is allowed to be asymmetric) as a compound pendulum. It is well-known from the pioneering work of Krein [8] that when purely imaginary eigenvalues collide in a Hamiltonian system, two types of behavior are possible: (i) if they satisfy Krein condition (i.e. they have same Krein signature) under perturbation they remain on the imaginary axis and (ii) if they have opposite Krein signature, then this leads to a linear instability under perturbation. Namely, this type of behavior was obtained and shown by us in Figure 2. In the literature the collision of purely imaginary eigenvalues in case (ii) is referred to as the 1:-1 resonance – Hamiltonian Hopf bifurcation. Note, that the 1:-1 resonances are generally non-semisimple, but exceptionally, may be semisimple.

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БИФУРКАЦИОННО ПОВЕДЕНИЕ НА ЕДНА ХАМИЛТОНОВА СИСТЕМА С ДВЕ СТЕПЕНИ НА СВОБОДА

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Ключови думи: *бифуркационно поведение, хамилтонова система, две степени на свобода*

Резюме: *Изследването на качествените промени в динамиката на една система е обект на бифуркационната теория. В тази статия ние анализираме бифуркационното поведение на една хамилтонова система с две степени на свобода описваща ездача и люлеещото се напомпване (от седнала позиция) като сложно махало. Нашите аналитични пресмятания предсказват, че бифуркация на Хамилтон-Хопф (1:-1 резонанс) има място.*