Mechanics Transport Communications

ISSN 1312-3823 (print)
ISSN 2367-6620 (online)
volume 16, issue 3/1, 2018

Academic journal http://www.mtc-aj.com **article № 1642**

COMPUTING OF NON-INTEGRABILITY WITH COMPUTER ALGEBRA METHODS

Georgi Georgiev
ggeorgiev@vtu.bg, ggeorgiev3@fmi.uni-sofia.bg

Todor Kableshkov University of Transport,
SU St. Kliment Ohridski, FMI,
BULGARIA

Key words: *Dyson potential, Hamiltonian systems, Non-integrability, Computer Algebra*

Abstract*. In this paper it is shown that the Hamiltonian system with Dyson potential is analytical non-integrable and formal non integrable. The Dyson system, appeared in 1962 by Dyson statistical physics research of the energy levels of one-dimensional Coulomb's gas. There is connection between the of system of point vortices - this is result by Calogero and Perelomov in 1978. Hamiltonian system has two integrals, that is why for integrability it is important the first nontrivial case* $n=3$*. Borisov and Kozlov in 1998 had proved that the system in case* $n=3$ *is non-integrable in analytical first integrals using splitting separarises method. In this paper it is proved the same statement, but way is different. Proof based of Duistermaat's idea when a system has a family of periodic solutions around equilibrium, and if the period function is infinitely branched, then if the system has additional analytical first integral - this integral is a constant. We call that the system with Hamiltonian H formally integrable if there exist formal power series* $\tilde{f}_1, \tilde{f}_2,..., \tilde{f}_n$ *in involution, where* $\tilde{f}_1, \tilde{f}_2,..., \tilde{f}_n$ *are functionally independent and Taylor expansion* $\tilde{H}$ *of H is a formal power series in* $\tilde{f}_1, \tilde{f}_2,..., \tilde{f}_n$*. An asymptotic behavior near equilibrium is like an integrable system. The formal integrability gives asymptotic information about the flow. For the proof of formal non-integrability in the paper is used Theory of Ziglin - Morales-Ruiz-Ramis. Computer algebra method is the Kovacic algorithm using a software for education, engineering and research Maple.*

1. Introduction. Let H be a smooth real-valued function of $2n$ real variables $(p,q)\in R^{2n}$. Let us also assume that $dH(0)=0$ where *0* is an equilibrium point for the Hamiltonian system X_H (with n degrees of freedom), given by

$$\frac{dq}{dt}=\frac{\partial H}{\partial p},\frac{dp}{dt}=-\frac{\partial H}{\partial q}$$

We often write the Hamiltonian systems in the form

$$\dot{x} = X_H(x), x \in R^{2n}$$

where X_H is the flow. The system is called Liouville - Integrable near *0* if there exist n functions in involution $f_1 = H, f_2, ..., f_n$ defined around 0, are functionally independent. The Poisson bracket of *f* and *g* are

$$\{f, g\} = \sum \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} = -\{g.f\}.$$

We say that the functions *f* and *g* are in involution if the Poisson bracket is commuteve. This means that $df_1, df_2, ..., df_n$ are linearly independent around the equilibrium *0* and $f_j = const$ for all *j* define smooth submanifolds, these level manifolds are invariant under X_{f_j} We have $X_{f_j} f_i = 0$ and $[X_{f_j}, X_{f_k}] = X_{\{f_j, f_k\}} = 0$ - these flows commute. The compact and connected component of $M_c := \{f_j = c_j, j = 1,..., n\}$ is diffeomorphed to a torus.
We call that the system with Hamiltonian *H* formally integrable if there exist formal power series $\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_n$ in involution, where $\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_n$ are functionally independent and Taylor expansion $\tilde{H}$ of *H* is a formal power series in $\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_n$. An asymptotic behavior near equilibrium is like an integrable system. The formal integrability gives asymptotic information about the flow. The functional independence of $\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_n$ is stronger than the independence of smooth functions $f_1, f_2, ..., f_n$ of which $\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_n$ are the Taylor series, because formal independence leads to a functional independence of a finite part of Taylor expansions.
We study the Hamiltonian system of *n* interacting particles of equal mass with a Hamiltonian

(1) $H = p_1^2 - p_1 p_2 + p_2^2 + \frac{3}{2} p_3^2 - \log|\sin q_1| - \log|\sin q_2| - \log|\sin(q_1 + q_2)|$.

Here $x_1, x_2, ..., x_n$ are the coordinates of particles, $y_1, y_2, ..., y_n$ are their momenta, and *V* is the potential energy in Dyson form.
The system with this potential was studied in [1] where the statistical properties of the energy levels of one dimensional Coulomb's gas are investigated. Calogero and Perelomov have found in [2] a connection between the system of point vortices and the Dyson system.

The function *V* is 2π-periodic (and even π periodic), so we can assume that the particles move in circles. The Hamiltonian system always has two integrals H , $F = \sum y_i$ The question of integrability of (1) was studied by Calogero and Perelomov [2] If *V* is a non-constant analytic periodic function without singularities, then the system (0.1) cannot be integrable for $n \geq 3$ (Kozlov [3]). Unfortunately Dyson potential has a real logarithmic singularity. Borisov and Kozlov in [4] have proved that the system in the case *n=3* is non-integrable in analytical first integrals.
We consider the same case *n=3*, but our approach is different.

Let us make (in the case *n=3*) canonical change of variables in (1)

$$y_1 = p_1 + p_3, y_2 = -p_1 + p_2 + p_3, y_3 = -p_2 + p_3,$$
$$q_1 = x_1 - x_2, q_2 = x_2 - x_3, q_3 = x_1 + x_2 + x_3.$$

We obtain

(2) $H = p_1^2 - p_1 p_2 + p_2^2 + \frac{3}{2} p_3^2 - \log|\sin q_1| - \log|\sin q_2| - \log|\sin(q_1 + q_2)|.$

The system has another integral

$$(3)\ F = 3p_3,$$

and has a stable equilibrium $p_1 = p_2 = 0, q_1 = q \ = \frac{\pi}{3}$.

The variable *q_3* is cyclic and it reduces the system (with *p_3=const*) to a system with *2*-degrees of freedom with Hamiltonian

2. Main result. Here we study the formal non-integrability of the system (1) in case *n=3*.

Our aim is the following:

Theorem 1.

a) The system with Hamiltonian (2) is not integrable by means of analytical first integral;

b) The system with Hamiltonian (2) is not formally integrable.

The motivation for proving formal non-integrability I received from the remarcable paper of J. J. Duistermaat [5] The Hamiltonian system with Dyson potential has similar structure.

Proof a):

Proposition 1. *On the manifold* $P := \{p_1 = p_2, q_1 = q_2\}$ *invariant under* X_H *the system with Hamiltonian (2) exists a family of periodical solutions around equilibrium*

$p_1 = p_2 = 0, q_1 = q \ = \frac{\pi}{3}$.

Proposition 2. *The period function has expression* $T(c) = \log\eta(c) + \Phi(c)$ *where* $c = \frac{1}{2e^E}$,

$\eta(c) = \varepsilon(B(c))\delta(B(c))$ *with*

$$B(c) = \frac{16(2/3)^{1/3} c^2}{(9c^2 - \sqrt{3}\sqrt{27c^4 - 256c^6})^{1/3}} + 2(2/3)^{1/3}(9c^2 - \sqrt{3}\sqrt{27c^4 - 256c^6})^{1/3}.$$

$\Phi(c)$ *is an analytical function of the variable c and*

$r_{1,2} = 1/2 - 1/2\sqrt{1+B} \pm 1/2\sqrt{2 - B + \frac{2}{\sqrt{1+B}}}$, $\varepsilon = 1/2(\arccos r_1 - \pi/3)$ *and*

$\delta = 1/2(-\arccos r_2 + \pi/3)$.

Proposition 3. *The system with Hamiltonian (2) does not possess any additional holomorphic first integral.*

Proof b):

Let us go back to the formal non-integrability. First we cosider the case when $K = H_2 + H_3$ i. e. when *K* is Taylor expansion up to order 3 in the variables

$\tilde{p}_1 := p_1, \tilde{p}_2 := p_2, \tilde{q}_1 := q_1 - \pi/3, \tilde{q}_2 := q_2 - \pi/3$ (we move the equilibrium to *0*)

$$K = \tilde{p}_1^2 - \tilde{p}_1 \tilde{p}_2 + \tilde{p}_2^2 + 4/3\tilde{q}^2 + 4/3\tilde{q}_1 \tilde{q}_2 + 4/3\tilde{q}_2^2 + 4/9\sqrt{3}\tilde{q}_1^2 \tilde{q}_2 + 4/9\sqrt{3}\tilde{q}_1 \tilde{q}_2^2.$$

Let us remote tildes. We obtain the Hamiltonian system

$$\dot{q}_1 = 2p_1 - p_2$$

$$\dot{q}_2 = -p_1 + 2p_2$$

$$\dot{p}_1 = -8/3q_1 - 4/3q_2 - 8/9\sqrt{3}q_1q_2 - 4/9\sqrt{3}q_2^2$$

$$\dot{p}_2 = -4/3q_1 - 8/3q_2 - 8/9\sqrt{3}q_1q_2 - 4/9\sqrt{3}q_1^2$$

We use the Theory of Morales-Ruiz- Ramis (see [7] for details) reducing the system to a Normal Variations Equations (NVE) near a non-trivial particular solution. We find a solution for $p := p_1 = p_2, q := q_1 = q_2$,

$$\dot{q}^2 = -8/9\sqrt{3}q^3 - 4q^2 + h.$$

Explicitly $q = \varphi(t) = -\frac{\sqrt{3}}{2} - \frac{3\sqrt{3}}{2}\wp(t, g_2, g_3)$, here $\wp$ is Weierstrass p-function with $g_2 = 4/3, g_2 = \frac{-4}{27}(h-2)$. Let we put $\eta_1 = dp_1, \eta_2 = dp$, $\xi_1 = dq_1, \xi_2 = dq_2$ then

$$\dot{\xi}_1 = 2\eta_1 - \eta_2$$

$$\dot{\xi}_2 = -\eta_1 + \eta_2$$

$$\dot{\eta}_1 = -(8/3 + 8/9\sqrt{3}\varphi)\xi_1 - (4/3 + 16/9\sqrt{3}\varphi)\xi_2$$

$$\dot{\eta}_2 = -(4/3 + 16/9\sqrt{3}\varphi)\xi_1 - (8/3 + 8/9\sqrt{3}\varphi)\xi_2.$$

If we get $\xi = \xi_1 - \xi_2$, then we find an equation for NVE

$\ddot{\xi} = (-8/3 + 4\wp(t, g_2, g_3))\xi$. This is a Lame-equation with $A=4,\ B = -8/3$ and in this case we have non Lame-Hermite solutions $A = n(n+1) \neq 4$ for $n \in Z$. We have non Briochi-Halphen- Crowford solutions $n + 1/2 \notin N$ and we have non the Baltassarri solutions $n + 1/2 \notin (1/3)Z \cap (1/4)Z \cap (1/5)Z - Z$ - the identity component of its Galois group is non-commutative (see [6]} for details).

The theory says that if the identity component of differential Galois group is non-commutative, then the system is not meromorphic integrable ([7]). This proves that in the case $H_2 + H_3$ there is no additional meromorphic (holomorphic) first integral.

Let us consider the case $H_2 + H_3 + H_4$. We get the system with particular solution for $p := p_1 = p_2, q := q_1 = q_2$ $q = \psi(t) = -\frac{3\sqrt{3}}{\sqrt{26}\sinh(2it) + 1}$ and $\eta_1 = dp_1, \eta_2 = dp$, $\xi_1 = dq_1, \xi_2 = dq_2, \xi = \xi_1 + \xi_2$ and we obtain for NVE

$\ddot{\xi} = (-4 + 8/9\sqrt{3}\psi(t) + 24\psi(t)^2)\xi$.

We need to algebrize NVE with a standard change of variable $w = \sqrt{26}\sinh(2it) + 1$.

Next we use the Kovacic algorithm with Maple to show that this equation has non Liouvillian solutions and the identity component of the Galois group of this equation is $SL(2, C)$. This means that the Hamiltonian system is non integrable with meromorphic first integrals. This proves that in the case $H_2 + H_3 + H_4$ there is no additional meromorphic (holomorphic) first integral.

The system H_2 is integrable, H_2+H_3 and $H_2+H_3+H_4$ are non integrable thats why we could conclude that the system $H_2+H_3+...+H_n$ is non integrable for each $n \geq 3$ ([8] for details). This proves b).

Referenses:

[1] Dyson F.J., (1962), Statistical theory of the enrgy levels of complex systems, I, II, III, J. Math Phys., **3**, 140-156, 157-165, 166-175.

[2] Calogero F., Perelomov A. M., (1978), Properties of Certain Matrices Related to the Equilibrium Configuration of the One-Dimensional Many-Body Problems with the Pair Potentials, $V_1(x) = -\log|\sin x|$ and $V_2(x) = \frac{1}{\sin^2(x)}$, Commun. math. Phys. **59**, no.12, 109-116.

[3] Kozlov V.V., (1995), Symmetries, Topology, and Resonace in Hamiltonian Mechanics, Izhevsk, Udmurt Univ. Moskow (in Russian).

[4] Borisov A. V., Kozlov V.V., (1998), Nonitegrability Of a System of Interacting Particles with the Dyson Potential, Doklady Akademii Nauk, Vol **306,** no. 1, 30--31.

[5] Duistermaat J. J., (1984), Non-integrability of the 1:1:2- resonance, Ergod. Th. \& Dynam. Sys., **4**, 553--568.

[6] Christov O., Georgiev G. (2015), On the integrability of a system describing the stationary solutions in Bose-Fermi mixtures, Chaos Solitons and Fractals, **77**, 8, 138--148.

[7] Morales-Ruiz J., (1999), Differential Galois Theory and Non-integrability of Hamiltonian Systems, Birkheauser.

[8] Yoshida H., (1988), Non-Integrability of the Truncated Toda Lattice Hamiltonian at Any Order, Commun. Math. Phys., **116**, 529--538.

ПРЕСМЯТАНЕ НА НЕИНТЕГРУЕМОСТ С МЕТОДИТЕ НА КОМПЮТЪРНИТЕ АЛГЕБРИ

Георги Георгиев

ggeorgiev@vtu.bg, ggeorgiev3@fmi.uni-sofia.bg

ВТУ Т. Каблешков, 1574, ул. "Гео Милев" 158, София

СУ Св. Кл. Охридски, ФМИ, София 1164, бул. Джеймс Баучър 5

БЪЛГАРИЯ

Ключови думи: *Потенциал на Дайсън, Хамилтонови системи, Неинтегруемост, Компютърни алгебри.*

Резюме: *В настоящата статия е изследвана задача произлязла от статистическата физика- хамилтонова система с потенциал на Дайсън. Доказана е неинтегруемост в аналитични първи интеграли и формална неинтегруемост в нетривиалния случай n=3. Използваните техники са от Коплексния анализ, Теорията на Зиглин-Моралес-Руиз за неинтегруемост, Алгоритъм на Ковачич с Maple.*