Mechanics Transport Communications

ISSN 1312-3823 (print)
ISSN 2367-6620 (online)
volume 15, issue 3, 2017

Academic journal http://www.mtc-aj.com **article № 1495**

DYNAMICS OF SWING OSCILLATORY MOTION IN HAMILTONIAN FORMALISM

Svetoslav Nikolov[1,2], Daniela Zaharieva[1]
S.Nikolov@imbm.bas.bg

[1]University of Transport, G. Milev Str. No 158, 1574 Sofia, BULGARIA; [2]Institute of Mechanics-BAS, Acad. G. Bonchev Str., Bl. 4, 1113 Sofia, BULGARIA

Key words: *dynamics, swing oscillatory motion, Hamiltonian formalism, homoclinic orbit*

Abstract: *in this paper we apply hamiltonian formalism to the analysis of dynamical behavior of swing oscillatory motion. In the swing system, the swinger is modeled (idealized) as a rigid dumbbell with three point masses, three lengths, an angular position with vertical and an angular position relative to the ropes. Under these assumptions, for asymmetrical (all masses and lengths are different) and symmetrical (two masses and two lengths are equal) cases the hamiltonian is obtained. For the symmetrical case, we detect the existence of a homoclinic orbit and present the equation for it.*

MECHANICS AND MATHEMATICS

1. INTRODUCTION

Dynamical systems can be separated into two classes – integrable and nonintegrable systems. A system from differential equations of the n^{th} order is completely integrable if it has n independent integrals of motion. According to the Liouville theorem, for Hamiltonian systems the existence of only $N = n/2$ integrals of motion is sufficient for integrability [1, 2]. Note, that it is only necessary for these integrals to be in involution, i.e, the Poisson brackets for any pair of them should be equal to zero.

For a completely integrable Hamiltonian system, its Hamiltonian $H(q, p)$ can be reduced in the form $H(P)$, where $P = P(q, p)$ is the generalized momentum vector and all its components are constants. It is well-known that the motion of an integrable Hamiltonian system is regular, i.e., it is periodic or quasi-periodic. By contrast, the nonintegrable Hamiltonian systems can be irregular (chaotic) [3, 4].

The discovery of irregular oscillations (chaotic behavior) in deterministic dynamical systems with different nature (mechanical, biological, chemical and economical), has become one of the most attractive area in science in recent 20^{ty} years [4, 5-7]. An important step towards the understanding of the global dynamics of a system of differential equations is the analysis of the existence of homoclinic/heteroclinic orbits (cycles). These orbits were first discovered by H. Poincare. In [8], he showed that the invariant manifolds of hyperbolic fixed points could cut each other at points – called homoclinics.

In dynamical systems with three (and higher) dimensions, the presence of a homoclinic orbit may imply the existence of chaotic behavior, horse shoes, and infinitely many nearby bifurcations, depending on the eigenvalues of the Jacobian matrix of the flow at the saddle point and on any symmetries that might be present in the system.

In Hamiltonian systems with more than one degree of freedom, the invariant manifolds of the hyperbolic periodic orbits may intersect (splitting) in the so-called homoclinic points. The presence of transversal homoclinic points implies the nonexistence of single-valued analytic integrals of motion independent of the Hamiltonian. Hence, the splitting of separatrices plays an important role in the creation of chaotic behavior [9, 10]. Note here, that in general the investigation of homoclinic points is quite complicated and involves some variant of perturbation methods.

For now little is known about homoclinic orbits in Hamiltonian systems with two and more than two degrees of freedom. In this paper we investigate a model of a swing pumped from the seated position. In [11], Case and Swanson modeled the rider and the swing as a compound pendulum, with a massive bob, m_1 at the rider's position on the seat and the rest of the body by two other bobs, m_2 that account for the extension of the body due to the rest of the body parts-arms, legs, head and so on. As shown in Fig. 1b, the two other bobs (for simplicity) of equal mass are represented as dumbbells that are positioned symmetrically about the scat of the swing. Since the model uses a symmetrical dumbbell, the center of mass of the swing is always at the position of the central mass, and therefore the parametric mechanism – periodically varying the center of mass relative to the pivot – does not apply. However, if the dumbbell is allowed to be asymmetric, then the center of mass will change and a parametric energizing mechanism does come into play. In [11], it is shown that the parametric mechanism dominates only as the amplitude becomes large.

In the full asymmetric dumbbell case the analysis becomes very complicated [12, 13]. This case is an idealization of pumping a swing while the rider is sitting down, as the effect produced only by rotation of the rider's body –see Fig. 1a. Using the notation in Fig. 1a, the kinetic T and potential energy U for the system are as follows:

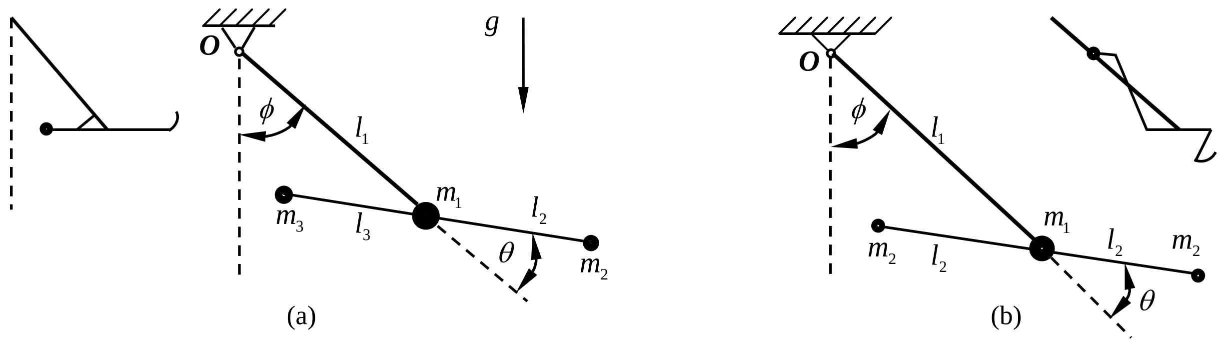

Figure 1. Person on a swing by point masses: a) the swinger is idealized as a rigid dumbbell with length $l_2 + l_3$, as all mass is located in point-masses $m_1 \neq m_2 \neq m_3$; b) the swinger is idealized as a rigid dumbbell with length $2l_2$ and a pair of point-masses.

(1)
$$T = \frac{1}{2}\left(I_1 + I_2 - 2l_1 N\cos\theta\right)\dot{\phi}^2 + \left(I_2 - l_1 N\cos\theta\right)\dot{\phi}\dot{\theta}^2 + \frac{1}{2}I_2\dot{\theta}^2,$$
$$U = -Ml_1 g\cos\phi + Ng\cos\left(\phi+\theta\right),$$

where l_1 is the length of massless rope of the playground swing, l_2 and l_3 are the lengths of legs and torso, ϕ and θ are the angles, and m_1, m_2 and m_3 are the dumbbell point-masses. The notations M, N, I_1 and I_2 are adopted from [11] and have the form

(2) $$M = m_1 + m_2 + m_3 \,,\; N = m_3 l_3 - m_2 l_2 \,,\; I_1 = M l_1^2 \,,\; I_2 = 2 m_2 l_2^2 \,.$$

Here we note that in the particular case when $l_2 = l_3$ and $m_2 = m_3$ (see Fig. 1b), then $N = 0$. This yields

(3) $$T = \frac{1}{2}(m_1 + 2m_2) l_1^2 \dot{\phi}^2 + m_2 l_2^2 \left(\dot{\phi} + \dot{\theta}\right)^2, \quad U = -(m_1 + 2m_2) l_1 g \cos\phi .$$

Our paper is organized as follows: In Section 2, we obtain the analytical results for the Hamiltonian of the models illustrated in Figure 1a and 1b. In Section 3, we present the analytical results for homoclinic loop of the model considered in Figure 1b – Eq. (3). Finally, Section 4 summarizes our results.

2. HAMILTONIAN FORM

To obtain the Hamiltonian $H(\phi, p_1, \theta, p_2)$ of the model from Fig. 1a (Eq. (2)), we firstly pass to the canonical momentum representation

(4) $$p_1 = \frac{\partial T}{\partial \dot{\phi}} \,,\; p_2 = \frac{\partial T}{\partial \dot{\theta}} \,.$$

This gives us the following expression for H

(5) $$H = T - U = \frac{1}{2\sqrt{\Delta}}\left(I_2 p_1^2 + a_1 p_2^2\right) - \frac{b_1}{\sqrt{\Delta}} p_1 p_2 - M l_1 g \cos\phi + N g \cos(\phi + \theta),$$

where

(6) $$a_1 = I_1 + I_2 - 2 l_1 N \cos\theta,\, b_1 = I_2 - l_1 N \cos\theta,\, \Delta = \left(a_1 I_2 - b_1^{\,2}\right)^2 .$$

The time-evolution of the system is then governed by equations of motion

(7) $$\dot{\phi} = \frac{\partial H}{\partial p_1} = \frac{1}{\sqrt{\Delta}}\left(I_2 p_1 - b_1 p_2\right), \quad \dot{\theta} = \frac{\partial H}{\partial p_2} = \frac{1}{\sqrt{\Delta}}\left(-b_1 p_1 + a_1 p_2\right),$$
$$\dot{p}_1 = -\frac{\partial H}{\partial \phi} = -M l_1 g \sin\phi + N g \sin(\phi - \theta), \quad \dot{p}_2 = N g \sin(\phi + \theta).$$

If $N = 0$, i.e. $m_2 = m_3$ and $l_2 = l_3$ (see Fig. 1b), then (5) and (7) have the form

(8) $$H = \frac{1}{2\psi_1}\left[p_1^2 + \left(1 + \frac{1}{2\psi_1 \psi_2}\right) p_2^2\right] - \frac{1}{\psi_1} p_1 p_2 + m_1 l_1 g (1 + 2\varepsilon) \cos\phi \,,$$

(9) $$\dot{\phi} = \frac{1}{\psi_1}\left(p_1 - p_2\right), \quad \dot{\theta} = \frac{1}{\psi_1}\left[- p_1 + \left(1 + \frac{1}{2\psi_1 \psi_2}\right) p_2\right],$$
$$\dot{p}_1 = m_1 l_1 g (1 + 2\varepsilon) \sin\phi \,, \quad \dot{p}_2 = 0 \,,$$

where $\psi_1 = m_1 l_1^2 (1+2\varepsilon), \psi_2 = m_1 l_1^2 \varepsilon\mu^2$, and $\varepsilon = \frac{m_2}{m_1} << 1$, $\mu = \frac{l_2}{l_1} << 1$ are small parameters.

In our considerations below we will investigate the model from Fig. 1b, i.e. Eqs. (8) and (9). It is seen that the Hamiltonian in (8) has no explicit time dependence and hence it is conserved. When the Hamiltonian is conserved, the phase space stratifies into distinct constant – H surfaces of dimension $2n-1$. On the other hand, the vanishing of $\dot{p}_2$ in (9) allows us to consider p_2 as a constant of the motion, $p_2 = c = const.$, thereby reducing the problem to a single degree of freedom, i.e.

$$\dot{\phi} = \frac{1}{\psi_1}(p_1 - c) \quad , \quad \dot{p}_1 = m_1 l_1 g (1+2\varepsilon)\sin\phi . \tag{10}$$

The fixed points of the system (10) are

$$\bar{\phi} = k\pi \quad , \quad \bar{p}_1 = c , \tag{11}$$

where $k = 0, \pm 1, \ldots$. In the equilibrium states the velocity is constant, c, and the potential has extreme. When k is an odd number, then $U(\bar{\phi})$ has maximum. When k is an even number, then $U(\bar{\phi})$ has minimum. In other words, for k odd number the fixed points are hyperbolic (saddles) and for k even number they are elliptic (centers). The first kind (saddle point) is associated to two invariant manifolds: the stable one W^s formed by all incoming orbits and the unstable one W^u composed of outgoing orbits. If the invariant manifolds coincide, then the so-called homoclinic connection (or another separatrix) takes place. Note that this configuration is valid for integrable systems. In all other cases, W^s and W^u do not coincide and can intersect along a homoclinic orbit: 1) *transversally* (at non zero angle) and non-*transversally* (homoclinic tangency). In our case, the separatrix is a phase trajectory which crosses the fixed point with coordinates $\left(\bar{\dot{\phi}} = \bar{p}_1 = c, \bar{\phi} = \pi\right)$, as the energy is $\bar{H} = E = \frac{1}{4\psi_1^2\psi_2} c^2 - m_1 l_1 g (1+2\varepsilon)$.

3. HOMOCLINIC SOLUTION OF (10)

In the previous section, we obtained some analytical results that we shall use in our calculations to find a homoclinic solution of (10). In our considerations, we introduced the small parameter ε.

To find the equation of the homoclinic orbit we use the Lindstedt perturbation method. For small perturbation parameter, ε, the system has the form (10). The homoclinic solution of system (10) satisfies the boundary conditions

$$\begin{aligned} &\phi \to \pi , \; p_1 \to c , \;\; for \quad t \to +\infty, \\ &\phi \to -\pi , \; p_1 \to c , \;\; for \quad t \to -\infty. \end{aligned} \tag{12}$$

The functions ϕ and p_1 as series in powers of ε can be presented in the form

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \ldots \quad , \quad p_1 = p_{10} + \varepsilon p_{11} + \varepsilon^2 p_{12} + \ldots \quad . \tag{13}$$

After substituting of (13) into (10) and accomplishing some transformations and analytical calculations we obtain the following equations for the functions θ_0, p_{10}

(14) $$\dot{\phi}_0 = \frac{1}{m_1 l_1^2}(p_{10} - c) \quad , \quad \dot{p}_{10} = -m_1 l_1 g \sin\phi_0 .$$

The function ϕ_0 is defined by the equation

(15) $$\ddot{\phi}_0 + \omega_0^{\,2} \sin\phi_0 = 0,$$

yielding the homoclinic solution

(16) $$\phi_0(t) = \pm 2 arctg(sh(\omega_0 t)),$$

where $\omega_0^{\,2} = g/l_1$. Hence, for function p_{10} we have

(17) $$p_{10}(t) = \pm m_1 l_1^{\,2} \frac{2\omega_0}{ch(\omega_0 t)} .$$

The solution (17) is also called a soliton (self-dependent wave) – see Fig. 2. It is well-known that solitons are nonlinear waves which do not change their speed and shape after a fully nonlinear interaction.

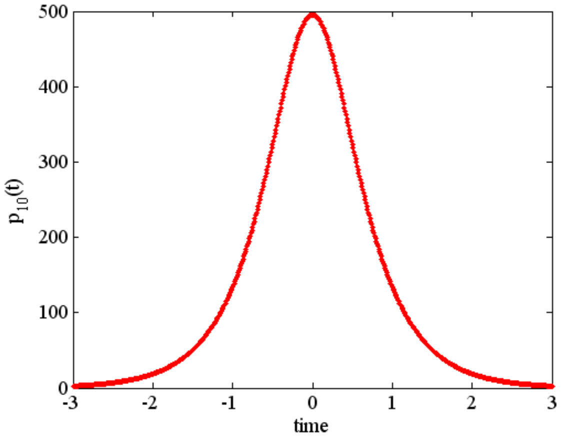

Figure 2. A soliton solution for the velocity of the separatrix Eq. (17) (in (ϕ_0, p_{10}) plane) when: $l_1 = 2.5[m]$, $m_1 = 20[kg]$ and time $t \in [-3,3]$ is in seconds. Note that when we have sign '+' in (17), the soliton motion is to the right (upper part of the separatrix).

4. CONCLUSION

It is well-known that the Hamiltonian presentation (formalism) of the dynamics is more perfect than the Lagrangian presentation (formalism). This circumstance is especially important in the theory of integrable systems and in the perturbation theory

Therefore, in this article, we used Hamiltonian formalism in exploring the dynamic behavior of the swing oscillatory motion. Two cases – asymmetrical and symmetrical – were considered. From the results obtained, it can be seen that in an asymmetrical case a complex (chaotic) behavior can occur. In a symmetrical case a homoclinic structure arises around which a periodic motion may occur.

References

[1] Whittaker E., A treatise on the analytical dynamics of particles and rigid bodies, Cambridge University Press, 1964.

[2] Arnold V., Mathematical methods in classical mechanics, Springer, Berlin, 1978.

[3] Guckenheimer J., Ph. Holmes, Nonlinear oscillations, dynamical systems, and bifurcations of vector fields, Springer, NY, 1996.
[4] Panchev S., Theory of Chaos. Sofia, Bulgarian Acad. Press, 2001.
[5] Nikolov S., O. Wolkenhauer, J. Vera, Tumors as chaotic attractors, Molecular BioSystems, 10(2), pp. 172-179, 2014.
[6] Nikolov S., N. Nedkova, Gyrostat model regular and chaotic behaviour, J. of Theoretical and Applied Mechanics, 45(4), pp. 15-30, 2015.
[7] Sprott J., Elegant chaos. Algebraically simple chaotic flows, World Scientific, Singapore, 2010.
[8] Poincare H., Les methodes nouvelles de la mecanique celeste, vol. 1-3, Gauthier-Villars, 1892; 1893; 1899.
[9] Gelfreich V., D. Sharomov, Examples of Hamiltonian systems with transversal homoclinic orbits, Physics Letters A, 197, pp. 139-146, 1995.
[10] Gelfreich V., Splitting of small separatrix loop near the saddle-center bifurcation in area-preserving maps, Physica D, 136, 266-279, 2000.
[11] Case W., M. Swanson, The pumping of a swing from the seated position, American J. of Physics, 58, pp. 463-467, 1990.
[12] Wirkus S., R. Rand, A. Ruina, How to pump a swing, College Mathematics Journal, 29, pp. 266-275, 1998.
[13] Linge S., An assessment of swinger techniques for the playground swing oscillatory motion, Computer Methods in Biomechanics and Biomedical Engineering, 15(10), pp. 1103-1109, 2012.

Acknowledgements

This work was supported by grant No 110/18.04.2017 of the University of Transport "T. Kableshkov" Sofia, Bulgaria.

ДИНАМИКА НА ЛЮЛЕЕЩОТО СЕ ОСЦИЛИРАЩО ДВИЖЕНИЕ В ХАМИЛТОНОВ ФОРМАЛИЗЪМ

Светослав Николов[1,2], Даниела Захариева[1]
S.Nikolov@imbm.bas.bg

[1]ВТУ „Т. Каблешков", ул. „Г. Милев" № 158, 1574 София,
[2]Институт по механика-БАН, ул. „Акад. Г. Бончев" бл. 4, 1113 София, БЪЛГАРИЯ

Ключови думи: *динамика, люлеещо се осцилиращо движение, Хамилтонов формализъм, хомоклинична орбита*

Резюме: *В тази статия използваме хамилтоновия формализъм за анализиране на динамичното поведение на люлеещото се осцилиращо движение. В люлеещата се система, люлеещият се е моделиран (идеализиран) като дъмбел с три точкови маси, три дължини, една ъглова позиция спрямо вертикалата и една относителна ъглова позиция спрямо въжето. При така направените приемания, за асиметричния (всички маси и дължини са различни) и симетричния (две маси и две дължини са равни) случаи е получен хамилтониана. За симетричния случай е открито съществуването на хомоклинична орбита и е представено нейното уравнение.*