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DETECTION OF A HOMOCLINIC ORBIT IN COMPOUND ELASTIC PENDULUM

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***Key words**: elastic pendulum, homoclinic orbit, nonintegrable Hamiltonian system*

***Abstract:** The simplest way to find complex (chaotic) behavior in a Hamiltonian system, e.g. as a starting point for consideration, is to look for homoclinic (heteroclinic) orbit(s).*

In this paper, under suitable assumptions, we detect the existence of a homoclinic orbit of a nonintegrable Hamiltonian system with two degrees of freedom – a compound elastic pendulum and present the equation for it.

1. INTRODUCTION

Nonlinear dynamics studies some conditions not observed in linear systems. Nonlinear behavior (movement and evolution) is typical for different dynamical systems. The investigation of nonlinear mechanical systems is an important and very active area [1]. For example, the pendulum is interesting as a paradigm of contemporary dynamics and, more importantly, since the differential equation of the pendulum is frequently encountered in various branches of modern mechanics [2, 9].

The study of complex phenomena in deterministic nonlinear dynamical system has attracted attention for applied scientists. This study poses two fundamental questions: 1) by what mechanisms does chaos occur and 2) how can one predict when chaos will occur in a specific dynamical system?

Dynamical systems can be separated into two main kinds: i) systems with conservation of phase volume and ii) systems with decrease of phase volume – called dissipative systems. Systems from kind (i) can be Hamiltonian and non-Hamiltonian systems. Systems are called Hamiltonian if their equations can be written in canonical form by means of a Hamiltonian $H(q,p,t)$, where q and p are generalized coordinates and momenta, and t is the time.

Generally, bifurcations can be local or global. Global bifurcations are qualitative change in the orbit structure of an extended region of phase space. Typical examples are homoclinic and heteroclinic bifurcations.

It is well-known that the loss of stability of periodic orbits in autonomous systems cannot always be reduced to investigation of bifurcation of fixed points of the Poincare map. Since in some cases the periodic orbit does not exist on the stability boundary, the Poincare map is not defined at the critical parameter value [3]. For such bifurcations, it is interesting to understand the structure of the limit set into which the periodic orbit transforms when the stability boundary is approached. Note, that such a limit set may be a homoclinic loop (orbit) to a saddle or to a saddle-note equilibrium state. In the “blue sky catastrophe” bifurcation scenario the periodic orbit approaches a set composed of a homoclinic bifurcation associated with the disappearance of the saddle-node equilibrium states and periodic orbits.

For mechanical systems which are modeled by Hamiltonian’s equations of motion, integrability is a central problem. Most Hamiltonian systems are not integrable but if a Hamiltonian dynamical system is integrable, then its first integrals provide a description of all features of the system [4, 5, 10].

It is well-known that all Hamiltonian systems with one degree of freedom and analytical Hamiltonian functions are integrable, and there is no splitting of separatrices. If separatrices of two saddles coincide to a form of heteroclinic orbit then we have a connection bifurcation. This is a global bifurcation as a necessary condition is that the two saddles have the same energy [6]. On the other hand, integrability is not a typical property of Hamiltonian systems with more than one degree of freedom. The phase spaces of such systems are difficult to visualize and a lower dimensional representation of the qualitative behavior of the system must be sought.

It is fairly well-known that Hamiltonian systems with two (or more) degrees of freedom are generally nonintegrable, which usually involves chaotic dynamics. All bounded trajectories of an integrable Hamiltonian system with $n=2$ degrees of freedom evolve in the $2n$ phase space on manifolds which are diffeomorphic (equivalent) to 2-tori [7]. If integrability is broken, KAM theory guarantees that tori (from resonances) will survive and the motion bears a nonregular (chaotic) character [8].

In this paper we consider a compound elastic pendulum (CEP) with two degrees of freedom- see Figure 1. For this pendulum the kinetic energy T and the potential energy U have the form

$$
\begin{aligned}
T &= \frac{m_1}{2} L^2 \dot{\theta}^2 + \frac{m_2}{2}\left[(l-\xi)^2 \dot{\theta}^2 + \dot{\xi}^2\right], \\
U &= -m_1 g L(1-\cos\theta) - m_2 g(l-\xi)(1-\cos\theta) + \frac{c}{2}\xi^2,
\end{aligned}
\tag{1}
$$

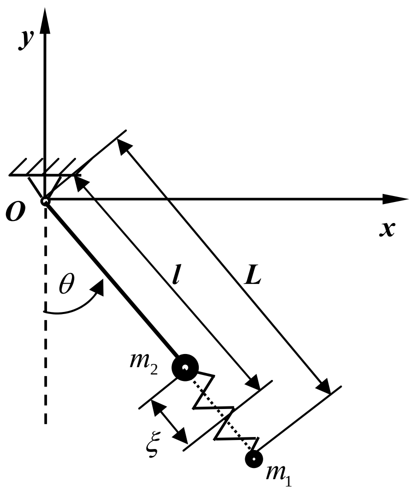

Figure 1. The mass-spring pendulum system.

where m_1 and m_2 are the masses, θ is the corresponding angle of the pendulum with the gravity vertical (defined $\bmod 2\pi$), ξ is the deviation of mass m_2 from equilibrium state, l is the distance from point O to equilibrium of mass m_2, L is the length of the pendulum, c is the spring constant and g is the acceleration of the gravity.

Here, we are interested in the dynamical behavior of the pendulum shown in Figure 1, and specially in the homoclinic orbit equation for it.

The paper is organized as follows: In Section two, we obtain the analytical results for the Hamiltonian of the compound elastic pendulum. In Section 3, we present the analytical results for homoclinic loop of CEP. Finally, Section 4 summarizes our results.

2. HAMILTONIAN FORM OF THE COMPOUND ELASTIC PENDULUM

In the Hamiltonian formulation, the state of a system is characterized not only by its positions q_i $(i = 1,...,n)$, but also by its momenta p_i. Thus, to obtain the Hamiltonian $H(\theta, p_1, \xi, p_2)$ of CEP, we firstly pass to the canonical momentum representation

$$p_1 = \frac{\partial T}{\partial \dot{\theta}} \quad , \quad p_2 = \frac{\partial T}{\partial \dot{\xi}} . \tag{2}$$

Hence, the Hamiltonian has the form

$$\begin{aligned} H(\theta, p_1, \xi, p_2) = T - U = \frac{1}{2m_2\psi}\left[m_2 p_1^2 + \psi p_2^2\right] + \\ + m_1 g L(1 - \cos\theta) + m_2 g(l - \xi)(1 - \cos\theta) - \frac{c}{2}\xi^2 , \end{aligned} \tag{3}$$

where $\psi = m_1 L^2 + m_2 (l - \xi)^2$. The time-evolution of the system is then governed by equations of motion

$$\begin{aligned} \dot{\theta} = \frac{\partial H}{\partial p_1}, \dot{\xi} = \frac{\partial H}{\partial p_2}, \\ \dot{p}_1 = -\frac{\partial H}{\partial \theta}, \dot{p}_2 = -\frac{\partial H}{\partial \xi}. \end{aligned} \tag{4}$$

An immediate consequence is

$$\dot{H} = \frac{\partial H}{\partial \theta}\dot{\theta} + \frac{\partial H}{\partial p_1}\dot{p}_1 + \frac{\partial H}{\partial \xi}\dot{\xi} + \frac{\partial H}{\partial p_2}\dot{p}_2 = 0, \tag{5}$$

the conservation of energy. Here we note that the system (4) is with two degrees of freedom and belongs to the class of nonintegrable Hamiltonian systems. The four-dimensional phase space $\mathrm{Z} = \{(\theta, \xi, p_1, p_2)\}$ is characterized by an extremely complex behavior of the phase trajectories and by existence of instability zones in which the motion has chaotic (nonregular) character [7, 8].

The fixed points of the system (4) in the (θ, ξ, p_1, p_2) plane defined as

$$\frac{\partial H}{\partial p_1} = \frac{\partial H}{\partial p_2} = \frac{\partial H}{\partial \theta} = \frac{\partial H}{\partial \xi} = 0 \tag{6}$$

are

$$p_1 = p_2 = 0 \, , \, \theta = k\pi \; (k = 0, \pm 1, \pm 2, ...), \xi = \pm \frac{m_2 g}{c} . \tag{7}$$

According to [8], the law of energy conservation $E = T - U$ defines an invariant three-dimensional hypersurface $H = E$ about the system phase flow. Here we are interested in a special phase curve- homoclinic orbit, lying in the three-dimensional manifold with level of energy E_h. This level corresponds to the 'upper' position (the masses standing upright vertically) of the compound elastic pendulum, i.e. $\theta = \pm\pi$, $\xi = -\frac{m_2 g}{c}$, $p_1 = p_2 = 0$. Hence, the manifold E_h has the form

$$H(\theta, p_1, \xi, p_2) = E_h = 2g\left[m_1 L + m_2\left(l + \frac{3}{4}\frac{m_2 g}{c}\right)\right]. \tag{8}$$

3. HOMOCLINIC SOLUTION

In the previous section, we obtained some analytical results that we shall use in our calculations so as to find a homoclinic solution in the three-dimensional manifold E_h. Note that this homoclinic solution is transversal intersection of the two-dimensional invariant surfaces W^s (stable manifold) and W^u (unstable manifold) of the hyperbolic fixed point $\theta = \pm\pi$, $\xi = -\frac{m_2 g}{c}$, $p_1 = p_2 = 0$. In our considerations, we introduce the small parameter $\varepsilon = {m_2}/{m_1} < 1$, i.e. $0 < m_2 < m_1$.

To find the equation of the homoclinic orbit we use the Lindstedt perturbation method. For small perturbation parameter, ε, the system (4) can be written in the form

$$\begin{aligned}
&\dot{\theta} = \frac{1}{m_1\left[L^2 + \varepsilon(l-\xi)^2\right]} p_1 \quad , \quad \dot{\xi} = \frac{1}{m_1\varepsilon} p_2 , \\
&\dot{p}_1 = -m_1 g \sin\theta\left[L + \varepsilon(l-\xi)\right], \\
&\dot{p}_2 = -\frac{\varepsilon(l-\xi)}{m_1\left[L^2 + \varepsilon(l-\xi)^2\right]^2} p_1^2 + m_1 g\varepsilon(1-\cos\theta) + c\xi .
\end{aligned} \tag{9}$$

The homoclinic solution of system (9) satisfies the boundary conditions

$$\begin{aligned}
&\theta \to \pi , \ \xi \to -\frac{m_2 g}{c} , \ p_1 \to 0 , \ p_2 \to 0 , \ \ for \quad t \to +\infty , \\
&\theta \to -\pi , \ \xi \to -\frac{m_2 g}{c} , \ p_1 \to 0 , \ p_2 \to 0 , \ \ for \quad t \to -\infty .
\end{aligned} \tag{10}$$

The functions θ, ξ, p_1 and p_2 as series in powers of ε can be presented in the form

$$\begin{aligned}
&\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \ldots \quad , \quad \xi = \xi_0 + \varepsilon\xi_1 + \varepsilon^2\xi_2 + \ldots , \\
&p_1 = p_{10} + \varepsilon p_{11} + \varepsilon^2 p_{12} + \ldots \quad , \quad p_2 = p_{20} + \varepsilon p_{21} + \varepsilon^2 p_{22} + \ldots .
\end{aligned} \tag{11}$$

After substituting of (11) into (9) and accomplishing some transformations and analytical calculations we obtain the following equations for the functions $\theta_0 , \zeta_1 , p_{10} , p_{22}$

(12)
$$\dot{\theta}_0 = \frac{1}{m_1 L^2} p_{10} \quad , \quad \dot{\xi}_1 = \frac{1}{m_1} p_{22}, \quad \dot{p}_{10} = -m_1 g L \sin\theta_0 ,$$
$$\dot{p}_{22} = \frac{1}{m_1 L^4} \xi_1 p_{10}^2 + 2\frac{m_1 g l^2}{L^2}\left(1-\cos\theta_0\right) + 2\frac{cl^2}{L^2}\xi_1 + c\xi_2 ,$$

where $\xi_2 = const$. Here we note that ξ_0, p_{20} and p_{21} are equal to zero.

The function θ_0 is defined by the equation

(13)
$$\ddot{\theta}_0 + \omega^2 \sin\theta_0 = 0,$$

yielding the homoclinic solution

(14)
$$\theta_0(t) = 2arctg\left(sh(\omega t)\right),$$

where $\omega^2 = g/l$. Hence, for function p_{10} we have

(15)
$$p_{10}(t) = m_1 L^2 \frac{2\omega}{ch(\omega t)} .$$

For the function ξ_1 respectively we have

(16)
$$\ddot{\xi}_1 = a(t)\xi_1 + b\left(1-\cos\theta_0\right) + \alpha ,$$

where $a(t) = \left(\frac{2\omega}{ch(\omega t)}\right)^2 + \frac{2cl^2}{m_1 L^2}$, $b = \frac{2gl^2}{L^2}$ and $\alpha = \frac{c}{m_1}\xi_2$.

The equation (16) will be investigated in our forthcoming work.

4. CONCLUSION

In this paper some analytical results for existence of a homoclinic loop in Hamiltonian systems with two degrees of freedom - compound elastic pendulum, were presented.

For Hamiltonian systems there are two types of hyperbolic fixed points: 1) saddle-focus and 2) saddle. If these systems are from fourth order, the homoclinic orbit lie in the transversal interaction of the stable W^s and unstable W^u manifolds of the origin. For both types of fixed points horseshoes exist on the level sets $H^{-1}(\varepsilon)$ for ε sufficiently small due to the structural stability of the horseshoes. Detecting of homoclinic (heteroclinic) orbits in a nonintegrable Hamiltonian system is not a easy task, but the existence of horseshoes forms conditions for the developing of chaotic behavior [11]. Note that our findings here should stimulate further numerical and analytical studies of system (9).

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ОТКРИВАНЕ НА ХОМОКЛИНИЧНА ОРБИТА В СЪСТАВНО ЕЛАСТИЧНО МАХАЛО

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БЪЛГАРИЯ

Ключови думи*: еластично махало, хомоклинична орбита, неинтегруема хамилтонава система*

Резюме: *Най-простият начин за откриване на комплексно (хаотично) поведение в една хамилтонова система е като се види дали съществува(т) хомоклинична (хетероклинична) орбита(и).*

В тази статия (в резултат на подходящи приемания), ние откриваме съществуването на хомоклинична орбита в неинтегруема хамилтонова система с две степени на свобода- сложно еластично махало, като предстяваме нейното уравнение.