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DYNAMIC BEHAVIOR OF AN ANGULAR RATE SENSOR MODEL

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Abstract: *In this paper we investigate the dynamical behavior of an angular rate sensor model when the unsymmetrical nonlinear restoring force is included. Our analytical calculations predict that angular velocity (directed along axis z) acts as a key parameter and the equilibrium states of the system can only lose their stability. This is confirmed by numerical simulations.*

1. INTRODUCTION

The study of instabilities plays a central role in the modern theory of dynamical systems In other words, we are interested in the qualitative changes in the dynamics at a change in the parameter values. The most simple and best studied case is the bifurcation of the equilibrium state in the presence of one parameter [1-3]. More complex bifurcations are described (examined) in the theory of bifurcations: bifurcation of the equilibrium state in the presence of more than one parameter in the system; bifurcation of periodic motion; relationship between the equilibrium and / or limit cycles; bifurcations of the more complex basis sets.

The gyroscope has a wide spectrum of applications in automotive, space engineering, military and aeronautical industry, medicine and so on. For these reasons, many research groups in Europe, USA and Asia have been investigating gyro architectures and technologies [4]. In recent years, the gyro (which measures angular rotation around a fixed axis with respect to an inertial space) is a key sensor in modern navigation systems [5].

Gyroscopic forces have two useful perspectives in the dynamics of mechanical systems: (i) they create coupling between different degrees of freedom, just like mechanical couplings; (ii) they rotate the velocity vector just like magnetic field acting on a charged particle. Note that gyroscopic forces are very useful in the stabilization of dynamical systems, because they are non-potential forces with zero power [4].

It is well-known that the operating principle of all vibrating gyros is based on the effect of Coriolis force $\vec{F}_C$ on a vibrating mass [5, 6]. In Figure 1, a simple model of vibrating angular rate sensor (as a two degree of freedom spring-mass-damper system) is shown

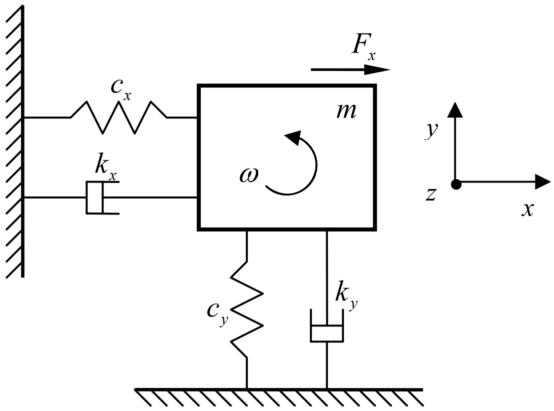


Figure 1. A simple model of vibrating angular rate sensor.

where m is the mass of the rotating reference frame, ω is the angular velocity (directed along axis z) of the reference frame, k_x and k_y are the damping coefficients along x and y axes, c_x and c_y are the spring constants along x and y axes and F_x is the excited force along axis x. According to [5, 6], such gyroscopes are frequently refered to as MEMS (Micro-Electro-Mechanical-Systems) gyroscopes. The Coriolis force (exercised) by a mass m moving in a rotating reference frame (see Fig. 1) is equal to

(1) $$\vec{F}_C = 2m(\vec{v} \times \vec{\omega}),$$

where $\vec{v}$ is the mass velocity in the rotating reference frame. Usually, the effect of the Coriolis force can be defined from dynamic equations describing the motion of the system shown in Figure 1. According to [5], the motion equations can be written in the form

(2) $$\begin{aligned} m\ddot{x} + k_x\dot{x} + c_x x - 2\omega m\dot{y} = F_x, \\ m\ddot{y} + k_y\dot{y} + c_y y + 2\omega m\dot{x} = 0. \end{aligned}$$

Here we note that the primary oscillating mode is excited when F_x is a harmonic (sinusoidal) force.

Most oscillating mechanical systems are not exactly linear but are approximately linear when the oscillations amplitude is small. In the case of a body on a spring, the restoring force F_R might actually have the form

(3) $$F_R = Cx + \Lambda x^3,$$

which is approximated by the linear formula $F_R = Cx$ - when the displacement x is small. The constant Λ is a measure of the strength of the nonlinear effect. It is well-known that if $\Lambda < 0$, then F_R is less than its linear approximation and the spring is said to be softening as x increases. Conversely, if $\Lambda > 0$, then the spring is hardening as x is increases. The formula (3) is typical of nonlinear restoring forces that are symmetrical about $x = 0$. If the restoring force is unsymmetrical about $x = 0$, the leading correction to the linear case will be a term in x^2, i.e.

(4) $$F_R = Cx + Bx^2.$$

where $C = [c_x, c_y]^{-1}$ and $B = [\beta_x, \beta_y]^{-1}$ are matrices.

Since an important performance parameter for a vibratory gyroscope is its zero rate output or zero bias, in this paper we assume that $k_x = k_y = F_x \approx 0$ and F_R has the form (4). Thus, the system (2) in normal form can be written as

$$\begin{aligned} \dot{y}_1 &= y_2 \,, \\ \dot{y}_2 &= -a_1 y_1 + a_2 y_4 - a_3 y_1^2 \,, \\ \dot{y}_3 &= y_4 \,, \\ \dot{y}_4 &= -a_2 y_2 - a_4 y_3 - a_5 y_3^2 \,, \end{aligned} \tag{5}$$

where

$$a_1 = \frac{c_x}{m} \quad , \quad a_2 = 2\omega \quad , \quad a_3 = \frac{\beta_x}{m} \quad , \quad a_4 = \frac{c_y}{m} \quad , \quad a_5 = \frac{\beta_y}{m} \,, \tag{6}$$

and

$$y_1 = x \quad , \quad y_2 = \dot{x} \quad , \quad y_3 = y \quad , \quad y_4 = \dot{y} \,. \tag{7}$$

Here we note that system (5) is a particular case of the general nonlinear system with two degrees of freedom (considered by us in [4]), where it is assumed that the forces in the right-hand sides are nonlinear from second order.

The paper is organized as follows: in Section 2 and 3 we present analytical and numerical results concerning the system (5) behavior. In Section 4 we discuss and summarize our results.

2. QUALITATIVE ANALYSIS

The steady (fixed points in the phase space) states of the system (5), $E = \left(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4\right)$, are found by equating the right-hand sides of (5) to zero. Thus, according to [4] they can be analytically estimated and are defined by the following set of algebraic equations, including the constants of the model:

$$\begin{aligned} &F_p^{(1)} : \bar{y}_1^{(1)} = \bar{y}_2^{(1)} = \bar{y}_3^{(1)} = \bar{y}_4^{(1)} = 0, \\ &F_p^{(2)} : \bar{y}_1^{(2)} = \frac{a_1}{a_3} \,, \bar{y}_3^{(2)} = -\frac{a_4}{a_5} \,, \bar{y}_2^{(2)} = \bar{y}_4^{(2)} = 0, \\ &F_p^{(3)} : \bar{y}_1^{(3)} = \bar{y}_2^{(3)} = \bar{y}_4^{(3)} = 0 \,, \bar{y}_3^{(3)} = -\frac{a_4}{a_5}, \\ &F_p^{(4)} : \bar{y}_1^{(4)} = \frac{a_1}{a_3} \,, \bar{y}_2^{(4)} = \bar{y}_3^{(4)} = \bar{y}_4^{(4)} = 0. \end{aligned} \tag{8}$$

In this paper we are interested in the behavior of the system (5) for fixed points $F_p^{(1)}$ to $F_p^{(4)}$. For these four fixed points, the divergence of the flow (5) is

$$D_4 = \frac{\partial \dot{y}_1}{\partial y_1} + \frac{\partial \dot{y}_2}{\partial y_2} + \frac{\partial \dot{y}_3}{\partial y_3} + \frac{\partial \dot{y}_4}{\partial y_4} = 0 \,, \tag{9}$$

i.e. the system (5) is a nonlinear conservative one.

The characteristic equation for fixed points $F_p^{(1)}$ to $F_p^{(4)}$ has the form:

$$\chi^4 + q\chi^2 + s = 0. \tag{10}$$

The equation (10) is biquadratic and can be solved exactly, i.e.

$$\chi^2 = \frac{1}{2m}\left(c_x + c_y + 4m\omega^2 \pm \sqrt{D}\right), \tag{11}$$

where $q = a_1 + a_4 + a_2^2 = \frac{1}{m}\left(c_x + c_y + 4m\omega^2\right)$, $s = a_1 a_4 = \frac{1}{m^2} c_x c_y$ and D is the discriminant.

A necessary and sufficient condition for all χ^2 s to be real is that the discriminant D into (11) is nonnegative

$$D = \left(c_x - c_y\right)^2 + 8m\omega^2\left(c_x + c_y + 2m\omega^2\right) \geq 0 . \tag{12}$$

Note, that the real χ^2 s are all nonpositive if and only if

$$q \geq 0 \quad , \quad s \geq 0. \tag{13}$$

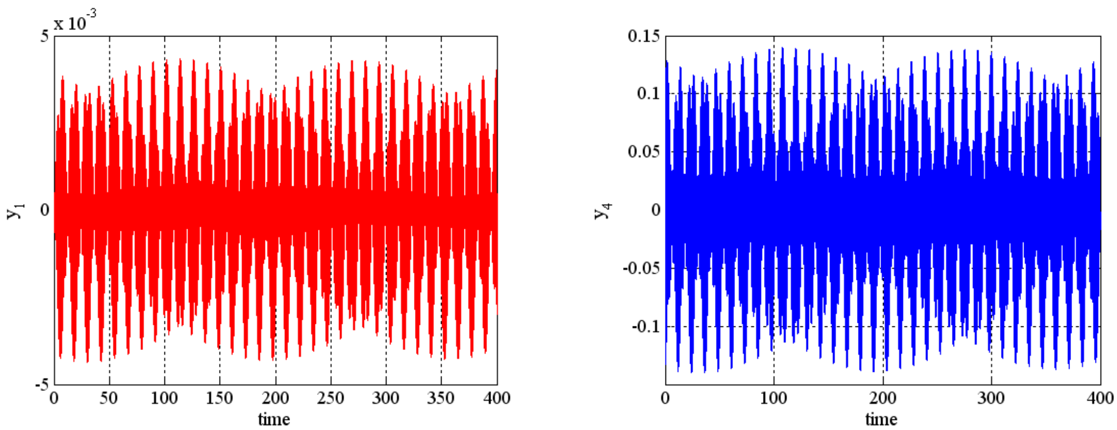
Inequalities (12) and (13) form a criterion for the eigenvalues χ to be purely imaginary, i.e. $\chi = \pm in$, where $n \geq 0$ is real (spectral) stability. This is however only a necessary condition for all solutions (of the linear form of (5)) to be bounded and thus to be marginally (linearly) stable [7]. According to [4, 8], here the second critical case is valid and the equilibrium states $F_p^{(1)}$ to $F_p^{(4)}$ can only lose their stability.

3. NUMERICAL ANALYSIS

The values chosen for the parameters and used in the numerical analysis are:

$$\begin{aligned} & m = 0.2[kg] \quad , \quad c_x = 200\left[N/m\right] \quad , \quad c_y = 150\left[N/m\right], \\ & \beta_x = 20\left[N/m^2\right] \quad , \quad \beta_y = 15\left[N/m^2\right] \quad , \quad \omega \in [10,100]\left[s^{-1}\right]. \end{aligned} \tag{14}$$

The dependence of the system's behavior (stable or unstable) on ω is shown in Figs. 2 and 3. We fix the model parameters (see (14)) and vary ω. It is seen that for smaller values of the angular velocity ω (i.e. $\omega = 10$) the system (5) has unstable solutions. These results are in accordance with the analytical results obtained in previous Section 2.
In addition, with increasing angular velocity, the magnitude of the real eigenvalues becomes smaller until they collide at the origin and form a zero eigenvalue of the algebraic multiplicity 2.



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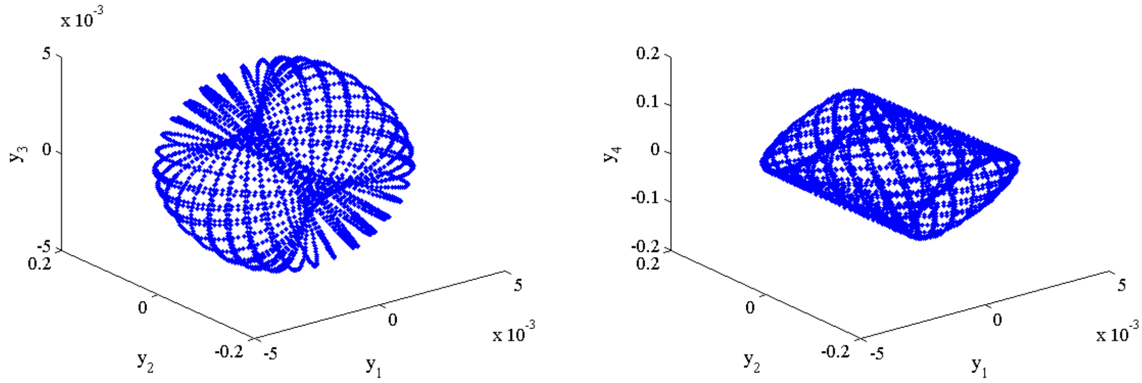

Figure 2. Unstable solutions and phase space of system (5) for $m=0.2$, $c_x=200$, $c_y=150$, $\beta_x=20$, $\beta_y=15$ and $\omega=10$ when $y_1(0)=y_3(0)=0$, $y_2(0)=y_4(0)=0.1$.

If ω grows further, the double eigenvalue splits into two purely imaginary eigenvalues - algebraic multiplicity 1 (i.e. simple) takes place and the equilibrium is marginally stable.

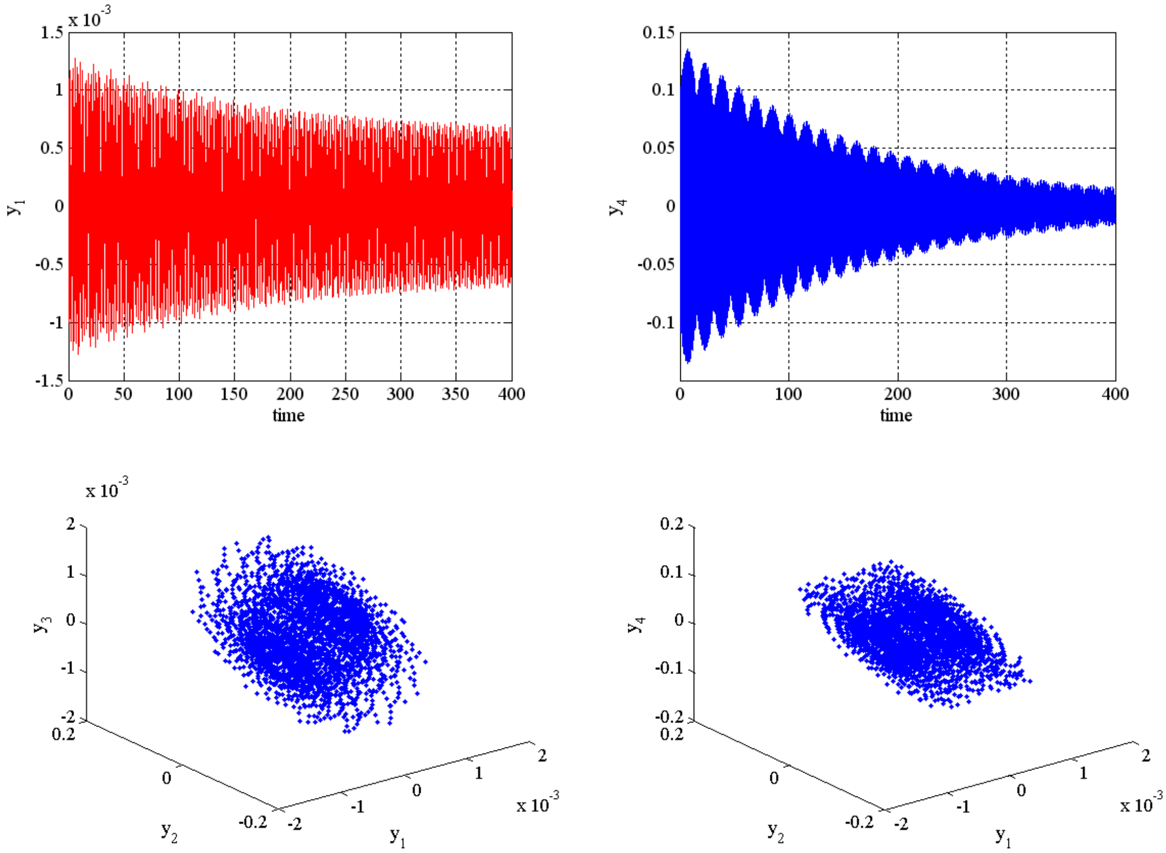

Figure 3. Stable solutions and phase space of system (5) for $m=0.2$, $c_x=200$, $c_y=150$ $\beta_x=20$, $\beta_y=15$ and $\omega=100$ when $y_1(0)=y_3(0)=0$, $y_2(0)=y_4(0)=0.1$.

4. CONCLUSION

The paper presents a study of the dynamical behavior of an angular rate sensor model, using analytical and numerical tools. Considering the case of an unsymmetrical nonlinear restoring force we obtain a particular case of the general nonlinear system with two degrees of freedom (considered by us in [4]), where it is assumed that the forces in the right-hand sides are nonlinear from second order. In Section 2 we find: 1) the necessary condition for all solutions of system (5) to be bounded and therefore to be marginally (linearly) stable as function of angular velocity ω and 2) that the equilibrium states $F_p^{(1)}$ to $F_p^{(4)}$ of system (5) can only lose their stability. In Section 3, we check the validity of our analytical results with numerical examples. Generalizing our results in Section 2 and 3, we conclude that angular velocity ω acts as a key parameter in the dynamical behavior of system (5).

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ДИНАМИЧНО ПОВЕДЕНИЕ НА ЕДИН МОДЕЛ НА ЪГЛОВО СКОРОСТЕН СЕНЗОР

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Ключови думи: *нелинейна динамика, МЕМС жироскопи, качествен и числен анализ*

Резюме: *В тази статия изследваме динамичното поведение на един модел на ъглово скоростен сензор, когато е включена несиметрична нелинейна възстановяваща сила. Аналитичните ни пресмятания предсказват, че ъгловата скорост около ос z действа като ключов параметър, а равновесните състояния на системата могат само да губят своята устойчивост. Това се потвърждава от числените симулации.*