Mechanics Transport Communications

Academic journal http://www.mtc-aj.com

ISSN 1312-3823 (print)
ISSN 2367-6620 (online)
volume 13, issue 3/2, 2015
article № 1216

ON THE HOLONOMY GROUP OF HYPERSURFACES OF SPACES OF CONSTANT CURVATURE

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Key words: *Space of constant curvature, hypersurface, holonomy group*

Abstract. *We classify hypersurfaces M^n of manifolds of constant nonzero sectional curvature according their restricted homogeneous holonomy groups. It turns out that outside of the evident cases (restricted holonomy group $SO(n)$ and flat submanifolds) only two cases arise: restricted holonomy group $SO(k) \times SO(n-k)$ (when M^n is locally a product of two space forms) and $SO(n-1)$ (when M^n is locally a product of an $(n-1)$ -dimensional space form and a segment).*

1. Introduction.

The holonomy groups are fundamental analytical objects in the theory of manifolds and especially in the theory of Riemannian manifolds. The holonomy group of a Riemannian manifold reflects for example on local reducibility of the manifold. In [6] M. Kurita classifies the conformally flat Riemannian manifolds according their restricted homogeneous holonomy group.

There exists a similarity between the conformal flat Riemannian manifolds and the hypersurfaces of a Riemannian manifold, see e.g. the remark of R. S. Kulkarni in [5]. So it is natural to look for a result in the submanifold geometry, analogous to the Kurita's theorem. In [3] S. Kobayashi proves that the holonomy group of a compact hypersurface of E^{n+1} is $SO(n)$. Generalizations of Kobayashi's result are obtained by R. Bishop [1] and G. Vranceanu [8].

In this paper we consider analogous question for hypersurfaces of non-flat real space forms according their holonomy groups. Namely we prove:

Theorem 1. *Let M^n $(n \geq 3)$ be a connected hypersurface of a space $\widetilde{M}^{n+1}(\nu)$ of constant positive sectional curvature ν. Then the restricted homogeneous holonomy group H_p of M^n in any point p is in general the special orthogonal group $SO(n)$. If H_p is not $SO(n)$ at any point $p \in M^n$, then one of the following cases appears:*

a) $H_p = SO(k) \times SO(n-k), 1 < k < n-1$ and M^n is locally a product of a k-dimensional space of constant sectional curvature $\nu + \lambda^2$ and an $(n-k)-$ dimensional space of constant sectional curvature $\nu + \mu^2$, with $\nu + \lambda\mu = 0$;

b) $H_p = SO(n-1)$ and M^n is locally a product of an $(n-1)$-dimensional space of *constant sectional curvature and a segment.*

A similar theorem for complex manifolds is proved in [7].

2. Preliminaries.

Let $\widetilde{M}^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold with metric tensor g and denote by $\widetilde{\nabla}$ its Riemannian connection. It is well known that if $\widetilde{M}^{n+1}$ is of constant sectional curvature ν , then its curvature operator $\widetilde{R}$ has the form

$$\widetilde{R}\,(x,y) = \nu\, x \wedge y$$

where the operator $\wedge$ is defined by

$$x \wedge y = g(y,z)x - g(x,z)y\,.$$

Such a manifold is denoted by $\widetilde{M}^{n+1}(\nu)$. Now let M^n be a hypersurface of $\widetilde{M}^{n+1}(\nu)$ and denote by ∇ its Riemannian connection. Then we have the Gauss formula

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)$$

for vector fields X, Y on M^n, where σ is a normal-bundle-valued symmetric tensor field on M^n, called the second fundamental form of M^n in $\widetilde{M}^{n+1}$. Let ξ be a unit normal vector field. Then the Weingarten formula is

$$\widetilde{\nabla}_X \xi = -A_\xi X$$

and the operator A_ξ is related to σ by

$$g(\sigma(X,Y),\xi) = g\big(A_\xi X, Y\big) = g\big(A_\xi Y, X\big)\,.$$

Suppose that we have fixed a normal vector field ξ . Then we shall write A instead of A_ξ. The equations of Gauss and Codazzi are given respectively by

$$R(X,Y) = \nu\, X \wedge Y + AX \wedge AY\,,$$

$$(\nabla_X A)Y = (\nabla_Y A)X\,,$$

R denoting the curvature operator of M^n.

It is known that Lie algebra of the infinitesimal holonomy group at a point p of a Riemannian manifold M is generated by all endomorphisms of the form

$$(\nabla^k R)(X,Y;V_1,\dots,V_k)\,,$$

where $X, Y, V_1, \dots, V_k \in T_pM$ and $0 \le k < +\infty$ [4] . Moreover if the dimension of the infinitesimal holonomy group is constant, this group coincides with the restricted homogeneous holonomy group [4].

3. Proof of Theorem 1.

Let p be an arbitrary point of M^n. We choose an orthonormal basis $e_1, \dots, e_n$ of T_pM , which diagonalize the symmetric operator A, i.e.

$$Ae_i = \lambda_i e_i \qquad i = 1, \dots, n.$$

Then by the equation of Gauss we obtain

$$R\big(e_i, e_j\big) = \big(\nu + \lambda_i\lambda_j\big)e_i \wedge e_j\,. \tag{3.1}$$

First we note that M^n cannot be flat at p . Indeed if M^n is flat, we obtain from (3.1) $\nu + \lambda_i\lambda_j = 0$ for all $i \neq j$. Since $n > 2$ this implies easily $\nu + \lambda_1^2 = 0$, and because of $\nu > 0$ this is a contradiction.

Since M^n is not flat at p, there exist $i \neq j$, such that $\nu + \lambda_i\lambda_j \neq 0$. Then (3.1) implies that $e_i \wedge e_j$ belongs to the Lie algebra h_p of H_p. As in [6] we denote by $SO[i_1, \dots, i_k]$ the subgroup of $SO(n)$, which induces the full rotation of the linear subspace, generated by $e_{i_1}, \dots, e_{i_k}$ and fixes the remaining vectors. Denote also by $so[i_1, \dots, i_k]$ the Lie algebra of $SO[i_1, \dots, i_k]$. Then according to the above argument H_p contains $SO[i,j]$.

If H_p contains $SO(n)$, then $H_p = SO(n)$, because the restricted homogeneous holonomy group H_p of a Riemannian manifold is a subgroup of $SO(n)$, see [2].

Let H_p is not $SO(n)$. Then there exist k, $2 \le k \le n-1$ and indices $i_1, \dots, i_k$, such that H_p contains $SO[i_1, \dots, i_k]$ but doesn't contain $SO[i_1, \dots, i_k, u]$ for $u \neq i_1, \dots, i_k$. Without loss of generality we can assume that H_p contains $SO[1, \dots, k]$, but does not contain $SO[1, \dots, k, u]$ for $u>k$.

Let us suppose that h_p contains $so[a, u]$ for some $a \epsilon \{1, \dots, k\}$ and $u \epsilon \{k+1, \dots, n\}$. Since

$$[e_b \wedge e_a, e_a \wedge e_u] = e_b \wedge e_u$$

it follows that the Lie algebra h_p contains $e_b \wedge e_u$ for $b=1,\dots,k$. Hence h_p contains $so[1,\dots,k,u]$, which is a contradiction.

Consequently h_p doesn't contain $so[a, u]$ for any $a=1,\dots,k;\ u=k+1,\dots,n$. Then (3.1) implies

$$(3.2) \qquad \nu + \lambda_a \lambda_u = 0 \qquad a = 1, \dots, k\,;\ u = k+1, \dots, n.$$

Hence, using $\nu \neq 0$, we obtain $\lambda_1 = \dots = \lambda_k$ and $\lambda_{k+1} = \dots = \lambda_n$. Denote $\lambda = \lambda_1$; $\theta = \lambda_{k+1}$. Then by (3.2) $\nu + \lambda\theta = 0$, $\lambda \neq 0$, $\theta \neq 0$ and it follows easily $\lambda \neq \theta$, $\nu + \lambda^2 \neq 0$, $\nu + \theta^2 \neq 0$.

In a neighborhood W of p we consider continuous functions $\Lambda_1, \dots, \Lambda_n$, such that for any point q in W the numbers $\Lambda_1(q), \dots, \Lambda_n(q)$ are the eigenvalues of A. Since $\nu + \lambda^2 \neq 0$, $\nu + \theta^2 \neq 0$, then in an open subset V of W containing p we have

$$\nu + \Lambda_a(q)\Lambda_b(q) \neq 0 \qquad a, b = 1, \dots, k\,;$$
$$\nu + \Lambda_u(q)\Lambda_v(q) \neq 0 \qquad u, v = k+1, \dots, n\,.$$

Hence H_q contains $SO[1,\dots,k]$ and $SO[k+1,\dots,n]$. Suppose that $\nu + \Lambda_a(q)\Lambda_u(q) \neq 0$ for some $a=1,\dots,k;\ u=k+1,\dots,n$. Then h_q contains $e_a \wedge e_u$, so as before h_q contains $so[1,\dots,k,u]$ and analogously h_q contains $so[n]$, which is not possible. So $\nu + \Lambda_a(q)\Lambda_u(q) = 0$. Hence as before we find

$$\Lambda_1(q) = \dots = \Lambda_k(q)\,,\ \ \Lambda_{k+1}(q) = \dots = \Lambda_n(q)\,.$$

Consequently in a neighborhood V of p there exist a number k and continuous functions Λ(q), Θ(q) such that $\Lambda(q) \neq \Theta(q)$ and

$$(3.3) \qquad \Lambda_1(q) = \dots = \Lambda_k(q) = \Lambda(q) \neq 0\,, \qquad \Lambda_{k+1}(q) = \dots = \Lambda_n(q) = \Theta(q) \neq 0$$

for $q \epsilon V$. Since M^n is connected, k is a constant on M^n. Consequently (3.3) holds on M^n. On the other hand using $\nu + \Lambda\Theta = 0$ and the fact that $k\Lambda + (n-k)\Theta =$ tr A is smooth we conclude that Λ and Θ are smooth functions on M^n. Define two distributions

$$T_1(q) = \{x \epsilon T_q(M) :\ Ax = \Lambda(q)x\}\,,$$
$$T_2(q) = \{x \epsilon T_q(M) :\ Ax = \Theta(q)x\}\,.$$

It follows directly that T_1 and T_2 are orthogonal and for $X, Y \epsilon T_1$, $Z, U \epsilon T_2$ we have

$$R(X,Y) = (\nu + \Lambda^2) X \wedge Y\,,$$
$$R(Z,U) = \frac{\nu}{\Lambda^2}(\nu + \Lambda^2) Z \wedge U\,,$$
$$R(X,Z) = 0\,.$$

We choose local orthonormal frame fields $\{E_1, \dots, E_k\}$ of T_1 and $\{E_{k+1}, \dots, E_n\}$ of T_2 and we denote

$$\nabla_{E_i} \mathrm{E}_j = \sum_{s=1}^{n} \Gamma_{ijs} E_s\,.$$

Then $\Gamma_{ijs} = -\Gamma_{isj}$ for all $i,j,s=1,\dots,n$, in particular $\Gamma_{ijj} = 0$. As before let $a, b, c\ \epsilon \{1, \dots, k\}$ and $u, v\ \epsilon \{k+1, \dots, n\}$. From the second Bianchi identity we have

$$(\nabla_a R)(E_b, E_u) + (\nabla_b R)(E_u, E_a) + (\nabla_u R)(E_a, E_b) = 0$$

and hence

$$E_u(\Lambda^2)E_a \wedge E_b + (\nu + \Lambda^2)\sum_{c=1}^{k}\{\Gamma_{buc}\, E_a \wedge E_c - \Gamma_{auc}E_b \wedge E_c\}$$

$$+(\nu + \Lambda^2) \sum_{v=k+1}^{n} \{\frac{\nu}{\Lambda^2}(\Gamma_{abv} - \Gamma_{ba\,v})E_u \wedge E_v + \Gamma_{uav}E_v \wedge E_b - \Gamma_{ubv}E_v \wedge E_a\} = 0\,.$$

Consequently we obtain

(3.4)
$$E_u(\Lambda^2) = (\nu + \Lambda^2)\{\Gamma_{aau} + \Gamma_{bbu}\}\,,$$
$$(\nu + \Lambda^2)\Gamma_{uva} = 0$$

for all $a \neq b$. Since $\nu + \Lambda^2 \neq 0$ we find $\Gamma_{uva} = 0$, so T_2 is parallel.

Let $n - k \geq 2$. Then analogously to the above T_1 is also parallel. Now (3.4) implies that Λ doesn't depend on E_u and analogously Θ doesn't depend on E_a. Hence, using $\nu + \Lambda\Theta = 0$ we conclude that Λ and Θ are constants. So we obtain the case a) of our Theorem.

Let $n - k = 1$. We shall show that under the assumption $H_p \neq SO(n)$ the distribution T_1 is again parallel. By the Codazzi equation we have

$$(\nabla_a A)(E_b) = (\nabla_b A)(E_a)\,.$$

This implies

$$E_a(\Lambda)E_b + (\Lambda - \Theta)\Gamma_{abn}E_n = E_b(\Lambda)E_a + (\Lambda - \Theta)\Gamma_{ban}E_n\,.$$

Hence $E_a(\Lambda) = 0$ for *a=1,…,n-1*. Now from

$$(\nabla_a A)(E_n) = (\nabla_n A)(E_a)$$

we obtain

$$E_n(\Lambda)E_a + (\Lambda - \Theta)\sum_{c=1}^{n-1}\Gamma_{anc}E_c = 0\,.$$

Hence we derive

(3.5)
$$E_n(\Lambda) = (\Lambda - \Theta)\Gamma_{aan}\,,$$
$$(\Lambda - \Theta)\Gamma_{acn} = 0 \quad for\ c \neq a\,.$$

Since $\Lambda \neq \Theta$ the last equality implies $\Gamma_{acn} = 0$ for $a \neq c$. On the other hand (3.5) implies $\Gamma_{aan} = \Gamma_{bbn}$. If $\Gamma_{aan} = 0$, then T_1 is parallel and from (3.5) $E_n(\Lambda) = 0$, so Λ is a constant. Because of $\nu + \Lambda\Theta \neq 0$ it follows that Θ is a constant too. Hence we obtain the case b) of our Theorem. Let us suppose that $\Gamma_{aan} \neq 0$. We compute directly

$$(\nabla_a R)(E_a, E_b) = (\nu + \Lambda^2)\Gamma_{aan}E_n \wedge E_b\,.$$

Hence $E_n \wedge E_b \epsilon h_p$ and as before it follows that $SO(n) = H_p$, which is not our case. This proves Theorem 1.

Remark. In the same way we can consider the case where $\widetilde{M}^{n+1}(\nu)$ is of constant negative sectional curvature ν . Then we obtain

Theorem 2. *Let M^n $(n \geq 3)$ be a connected hypersurface of a space $\widetilde{M}^{n+1}(\nu)$ of constant negative sectional curvature ν. Then the restricted homogeneous holonomy group H_p of M^n in any point p is in general the special orthogonal group $SO(n)$. If H_p is not $SO(n)$ at any point $p \in M^n$, then one of the following cases appears:*

a) $H_p = SO(k) \times SO(n-k), 1 < k < n-1$ and M^n is locally a product of a k-dimensional space of constant sectional curvature $\nu + \lambda^2$ and an $(n-k)-$ dimensional space of constant sectional curvature $\nu + \mu^2$, with $\nu + \lambda\mu = 0$;

b) $H_p = SO(n-1)$ and M^n is locally a product of an $(n-1)$-dimensional space of constant sectional curvature and a segment.

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ВЪРХУ ГРУПАТА НА ХОЛОНОМИЯ НА ХИПЕРПОВЪРХНИНИ НА ПРОСТРАНСТВА С ПОСТОЯННА КРИВИНА

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Ключови думи: *пространства с постоянна кривина, хиперповърхнини, група на холономия.*

Резюме. *В тази статия правим класификация на хиперповърхнините M^n на пространства с постоянна ненулева секционна кривина според техните стеснени хомогенни групи на холономия. Доказваме, че освен очевидните случаи (стеснена група на холономия $SO(n)$ и плоски подмногообразия) се появяват само два случая: стеснена група на холономия $SO(k) \times SO(n-k)$ (когато M^n локално е произведение на две пространствени форми) и $SO(n-1)$ (когато M^n локално е произведение на $(n-1)$ – мерна пространствена форма и линия).*