**Mechanics Transport Communications**

ISSN 1312-3823 (print)
ISSN 2367-6620 (online)
volume 13, issue 3/2, 2015

*Academic journal* http://www.mtc-aj.com article № 1212

# NONLINEAR DYNAMICS OF A FLUID GYROSCOPE

**Svetoslav Nikolov[1,2], Nataliya Nedkova[1]**
S.Nikolov@imbm.bas.bg

*[1]University of Transport, G. Milev Str. No 158, 1574 Sofia,*
*[2]Institute of Mechanics-BAS, Acad. G. Bonchev Str., Bl. 4, 1113 Sofia*
*BULGARIA*

***Key words:*** *nonlinear dynamics, fluid (liquid) gyroscope, qualitative and numerical analysis*

***Abstract:*** *The article examines the dynamic behavior of a system of three nonlinear Ordinary Differential Equations describing the behavior of a liquid gyroscope. Our analysis and simulations (with specific choice of parameters) demonstrates that the equilibrium states are three – of saddle and saddle-focus type, and the oscillations are conditionally divided into two levels – "macro" and "micro" level. Changing the parameter* $f_3$ *leads to modification of the period of the oscillations of the "macro" level.*

## 1. INTRODUCTION

In recent years, increasing attention has been focused on the nonlinear dynamics of mechanical systems, including those associated with rigid body motion (gyroscope) [1-4]. One way to identify the dynamic changes of a system, and the strategies that may be used to address those changes, is to build models and explore their bifurcation behavior.

Chaos theory has been developed and chaos thoroughly studied over the past two decades. A chaotic system is a nonlinear deterministic system that displays complex, noisy-like and unpredictable behavior. The sensitive dependence upon an initial condition and on the system's parameter variations is a prominent characteristic of chaotic behavior [5, 6].

Dissipative systems are a special class of dynamical systems. In general, dissipative mechanical systems in more than two dimensions have bounded and unbounded orbits depending on the energy. The bounded trajectories of these systems do not converge to an equilibrium point or to a periodic or quasi-periodic orbit. In this case the flow is essentially aperiodic. A dynamical system is dissipative, if its phase volume contracts continuously, i.e $D_i < 0 \quad (i = 1,..., n)$. Only dissipative dynamical systems have attractors.

The idea for description of real hydrodynamic phenomena was firstly well developed by 1974, by Obukhov [7, 8]. It will be mentioned that Obukhov is the first who used the concept "system of hydrodynamic type" for square-nonlinear dynamic system with small number of degrees of freedom. This system has similar hydrodynamic invariants (invariants about the energy), which save the phase volume [9].

Let us consider the equation of a vortex

$$\frac{\partial}{\partial t}\nabla^2\psi + J\left(\psi, \nabla^2\psi\right) = 0, \tag{1}$$

where $J\left(\psi, \nabla^2\psi\right) = \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y}$ is a horizontal Jacobi operator for two-dimensional un-viscosity flow of homogeneous unbended fluid in the plane $\left(x, y\right)$. Here the function of current $\psi$ is bi-periodic for any time *t*, i.e.

$$\psi\left(x, y, t\right) = \psi\left(x + \frac{2\pi}{k}, y + \frac{2\pi}{l}, t\right). \tag{2}$$

In this domain, the functions of the Laplace operator $\nabla^2\psi = \lambda\psi$ are trigonometric about the argument $\left(mkx + nly\right)$, where $m, n \in Z$ and $\lambda = m^2k^2 + n^2l^2$.

The trigonometric functions can be used as a basis in the field spectral decomposition of $\psi$. Hence we have

$$\psi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[a_{mn}\cos\left(mkx + nly\right) + b_{mn}\sin\left(mkx + nly\right)\right], \tag{3}$$

where $a_{mn}\left(t\right)$ and $b_{mn}\left(t\right)$ are unknown functions. After substitution of (3) into (1) and accomplishing integration, we obtain the flow equations in different domains. These operations are well-known as Galerkin transformations. Thus, the infinity dimensional system from ordinary differential equations for spectral coefficients $c_{mn} = \left(a_{mn}, b_{mn}\right)$ is obtained. It is seen that the new system is a square-nonlinear, i.e.

$$\dot{c}_{mn} = \frac{dc_{mn}}{dt}\sum_{m_1}\sum_{n_1}\sum_{m_2}\sum_{n_2}k\left(m, n, m_1, n_1, m_2, n_2\right)c_{m_1n_1}c_{m_2n_2}, \tag{4}$$

where $k\left(m, n, m_1, n_1, m_2, n_2\right)$ are coefficients of nonlinear interaction. Here, we note that for a real investigation of system (4),a finite number of coefficients must be taken, as the minimal number is three. Hence, the nonlinear interaction between different components in (3) can be "caught".

Consider, for example, $\psi = a\cos ly + f\cos kx + g\sin ly\sin kx$. Thus, we have

$$\begin{aligned}
\dot{A} &= -\left(\frac{1}{k^2} + \frac{1}{k^2 - l^2}\right)klFG = pFG,\\
\dot{F} &= \left(\frac{1}{l^2} - \frac{1}{k^2 + l^2}\right)klAG = qAG,\\
\dot{G} &= -\left(\frac{1}{l^2} - \frac{1}{k^2}\right)klAF = rAF,
\end{aligned} \tag{5}$$

where $p + q + r = 0, A = -l^2a, F = -k^2f$ and $G = \frac{1}{\sqrt{2}}\left(k^2 + l^2\right)g$ are the spectral coefficients of $\nabla^2\psi$. The equations (5) have minimal number of degrees of freedom, which are necessary for the investigation of (1). Also, the equations (5) can be integrated exactly in Jacobian elliptical functions. According to [5, 9], the specific model exists, for which all particular solutions can be described from the three-fence (5) – triple. The free fluid motion here is inside of an ellipsoid with linear velocities. In this case, the form of the equations is equivalent with those of Euler for rigid body motion with one fixed point (gyroscope) [10]. Hence, the three-fence (5) is called hydrodynamic (fluid) gyroscope [5, 6, 9].

In this paper we consider the set of coupled equations

$$
\begin{aligned}
\dot{x} &= -bx + ayz + f_1, \\
\dot{y} &= -by + cxz + f_2, \\
\dot{z} &= -bz + dxy + f_3,
\end{aligned}
\tag{6}
$$

for real functions *x(t)*, *y(t)* and *z(t)*, where the overdot denotes differentiation with respect to the time-like independent variable *t*, and the coefficients *a* to *d* and $f_i\ (i = 1 \div 3)$. When the relations

$$
a + c + d = 0,\ ad > 0,\ ac < 0,\ cd < 0,\ b > 0,\ f_i \geq 0\,, \tag{7}
$$

are valid, the system (6) presents the motion of a fluid gyroscope with dissipation [5, 9]. Here, we give an existence of the results obtained in [5, 6, 9].

The paper is organized as follows: in Section 2 and 3 we present analytical and numerical results concerning the system (6) behavior for different values of bifurcation (control) parameters $f_i$. In Section 4 we discuss and summarize our results.

## 2. QUALITATIVE ANALYSIS

In this section, we investigate the system (6), which presents an autonomous nonlinear 3D dynamical model. According to [6], for $f_1 > 0,\ f_2 > 0$ and $f_3 > 0$, the equilibrium (steady state) points of system (6) are:

$$
\begin{aligned}
&\bar{x} = \frac{1}{b}\left(a\bar{y}\bar{z} + f_1\right), \quad \bar{y} = \frac{cf_1\bar{z} + bf_2}{b^2 - ac\bar{z}^2}, \\
&\bar{z}^5 - \frac{f_3}{b}\bar{z}^4 - \frac{2b^2}{ac}\bar{z}^3 + \frac{2b^2 f_3 - acdf_1 f_2}{abc}\bar{z}^2 + \frac{b^4 - cdf_1^2 - adf_2^2}{a^2c^2}\bar{z} - \frac{f_3 b^3 + bdf_1 f_2}{a^2c^2} = 0.
\end{aligned}
\tag{8}
$$

In order to determine the character of the fixed points (Eq. (8)) we make the following substitutions into (7)

$$
x = \bar{x} + x_1, \quad y = \bar{y} + x_2, \quad z = \bar{z} + x_3. \tag{9}
$$

Hence, after some transformations the system (6) has the form

$$
\begin{aligned}
\dot{x}_1 &= -bx_1 + c_1x_2 + c_2x_3 + ax_2x_3, \\
\dot{x}_2 &= -bx_2 + c_3x_1 + c_4x_3 + cx_1x_3, \\
\dot{x}_3 &= -bx_3 + c_5x_1 + c_6x_2 + dx_1x_2,
\end{aligned}
\tag{10}
$$

where

$$
c_1 = a\bar{z}, \quad c_2 = a\bar{y}, \quad c_3 = c\bar{z}, \quad c_4 = c\bar{x}, \quad c_5 = d\bar{y}, \quad c_6 = d\bar{x}. \tag{11}
$$

According to [11], the Routh-Hurwitz conditions for stability of fixed points (8) can be written in the form

$$
p = 3b > 0, \tag{12}
$$

$$
q = 3b^2 - c_1c_3 - c_2c_5 - c_4c_6 = 3b^2 - cd\bar{x}^2 - ad\bar{y}^2 - ac\bar{z}^2 > 0, \tag{13}
$$

$$
\begin{aligned}
r &= b^3 - c_1c_4c_5 - c_2c_3c_6 - b\left(c_1c_3 + c_2c_5 + c_4c_6\right) = \\
&= b^3 - 2acd\bar{x}\bar{y}\bar{z} - b\left(cd\bar{x}^2 + ad\bar{y}^2 + ac\bar{z}^2\right) > 0
\end{aligned}
\tag{14}
$$

$$
R = pq - r = 2\left[4b^3 + acd\bar{x}\bar{y}\bar{z} - b\left(cd\bar{x}^2 + ad\bar{y}^2 + ac\bar{z}^2\right)\right] > 0.
$$

Here the notations $p, q, r$ and $R$ are taken from [11]. The divergence of the flow (11) is

$$D_3 = \frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} + \frac{\partial \dot{x}_3}{\partial x_3} = -3b. \qquad (15)$$

The system (11) is always dissipative and has attractor because $D_3 < 0$ for $b > 0$.

The characteristic equation of the system (11) (which is equivalent of (6)) can be written as

$$\chi^3 + p\chi^2 + q\chi + r = 0. \qquad (16)$$

For

$$a = 0.095, b = 0.003, c = -0.1, d = 0.005, f_1 = 0.01, f_2 = 15.5, \ f_3 = 0.5 \qquad (17)$$

the number of equilibriums and their eigenvalues are given by:

$O_1 = (22.0357, 0.0845, 6.9931)$, then $(\chi_1, \chi_2, \chi_3) = (0.1812 \pm 1.1835i, -0.3713)$ - *saddle-focus*

$O_2 = (-29.3098, 0.1961, -5.2574)$, then $(\chi_1, \chi_2, \chi_3) = (0.0002 \pm 1.6532i, -0.0094)$ - *saddle-focus*

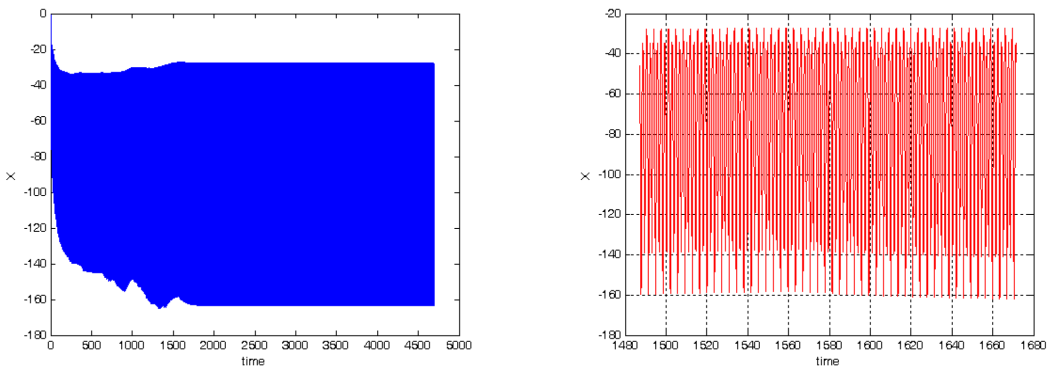
$O_3 = (-0.0004, 5136.7, 0)$, then $(\chi_1, \chi_2, \chi_3) = (-112.6079, 112.6019, -0.003)$ - *saddle*

These fixed points can be included in homoclinic and heteroclinic structures with two and three equilibriums, where their invariant manifolds $W^S$ and $W^U$ are meeting each other in a most intricate manner.

## 3. NUMERICAL ANALYSIS

In the previous section, we have obtained and shown some analytical results that we shall use in our numerical analysis of system (6). The corresponding values of the dimensionless system parameters are as those in (17), where $f_3 = 0.4; 0.51$. The governing equations of system (6), were solved numerically using MATLAB [12].

In Figures 1 and 2 the solutions of system (6) are shown. For $f_3 = 0.4$ the system has monotone "macro" dynamics (see the left panels) and periodic (with period one and two) "micro" dynamics (see the right panels). On the other hand, for $f_3 = 0.51$ the system has periodic "macro" dynamics with period one, as the form of the "micro" dynamics is the same with the case for $f_3 = 0.4$.



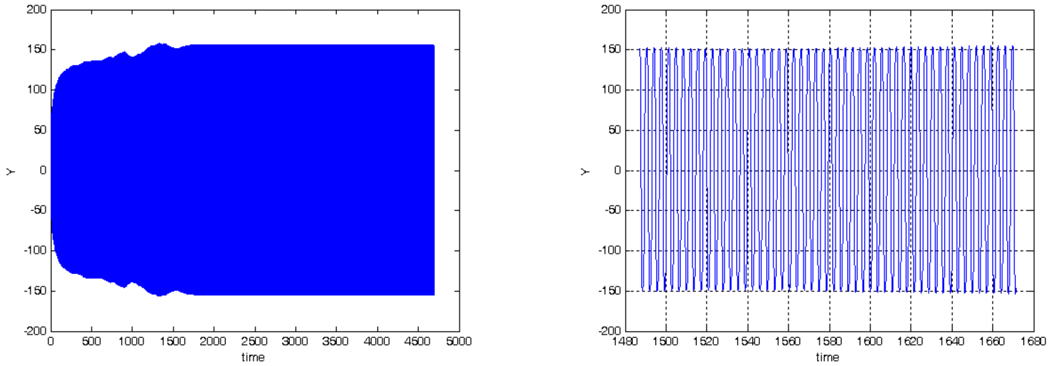



Figure 1. Macro (left panels) and micro (right panels) dynamics of system (6) for $a=0.095$, $b=0.003$, $c=-0.1$, $d=0.005$, $f_1=0.01$, $f_2=15.5$, $f_3=0.4$.

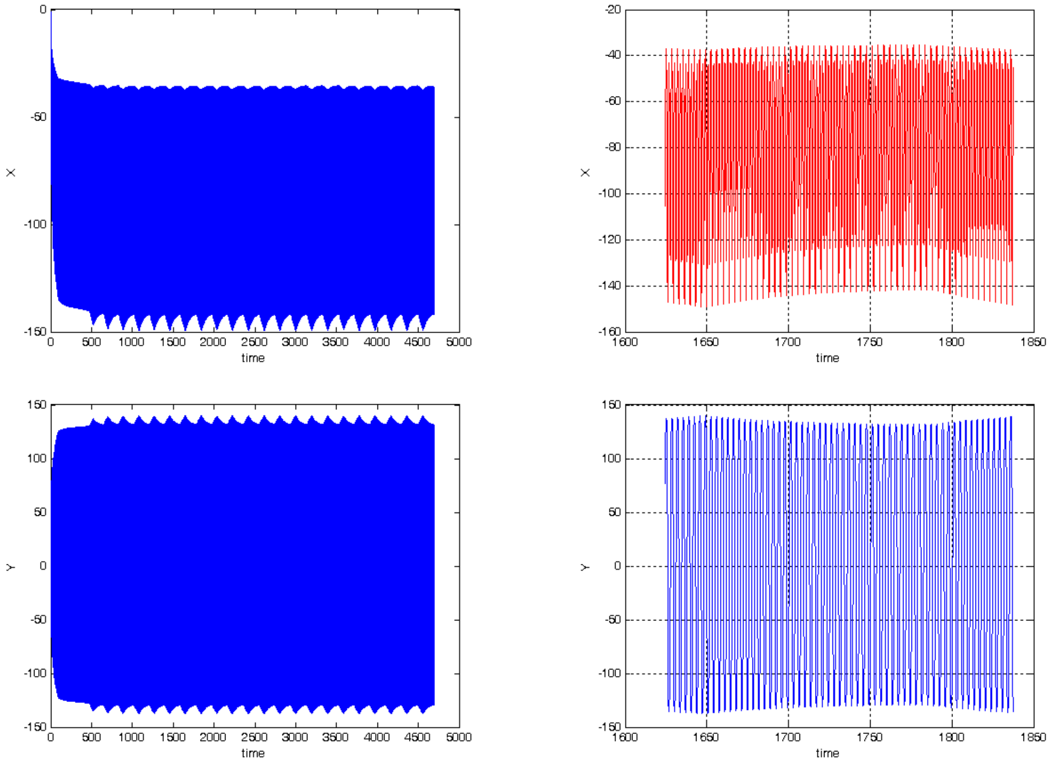



Figure 2. Macro (left panels) and micro (right panels) dynamics of system (6) for $a=0.095$, $b=0.003$, $c=-0.1$, $d=0.005$, $f_1=0.01$, $f_2=15.5$, $f_3=0.51$.

From dynamical point of view, as $f_3$ increases, the fixed points $O_1 - O_3$ change their position in the phase system space and the connection (meeting) between their invariant manifolds is different.

## 4. CONCLUSION

In this paper we have analyzed a 3-dimensional fluid gyroscope model. The focus has been on the dynamical properties and the role of the positive parameters $f_i$ in unstable parameter regions. In all simulations the initial conditions were $x_0 = y_0 = z_0 = 0.1$. Simulations suggest that when $f_3$ is a bifurcation parameter, the system has three fixed points of saddle-focus and saddle type. These fixed points change their position in phase space and the "macro" dynamics of the system (6) respectively.

Regarding the wide subject under consideration and the number of arising questions, the proposed investigation is a step to the profound and full analysis of the fluid gyroscope system.

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**Acknowledgements**

This work was supported by grant No 1616/22.04.2014 of the University of Transport "T. Kableshkov" Sofia, Bulgaria.

## НЕЛИНЕЙНА ДИНАМИКА НА ТЕЧЕН ЖИРОСКОП

**Светослав Николов[1,2], Наталия Недкова[1]**
S.Nikolov@imbm.bas.bg

***[1]ВТУ „Т. Каблешков", ул. „Г. Милев" № 158, 1574 София,***
***[2]Институт по механика-БАН, ул. „Акад. Г. Бончев" бл. 4, 1113 София,***
***БЪЛГАРИЯ***

***Ключови думи:*** *нелинейна динамика, течен жироскоп, качествен и числен анализ*

***Резюме:*** *В тази статия беше изследвано динамичното поведение на система от три нелинейни ОДУ описващи поведението на течен жироскоп. От извършения анализ и симулации (за специфичен избор на параметрите) става ясно, че равновесните състояния са три - от тип седло и седло-фокус, а трептенията са условно разделени на две нива – „макро" и „микро". Промяната на параметъра* $f_3$ *води до изменение на периода на трептенията на „макро" ниво.*