Mechanics Transport Communications ISSN 1312-3823
Academic journal issue 3, 2007
article № 0191
http://www.mtc-aj.com

ESTIMATES OF KHINTCHINE CONSTANT OF THE LORENTZ SEQUENCE SPACE

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Abstract: *We estimate the Khintchine constant, also called type-n constant of l_q. As corollaries we get estimates for some special cases of parameters, in particular we estimate the Jordan-von Neumann constant of Lorentz sequence space.*
Key words: *Khintchine constant, Lorentz sequence space.*

Let X be a Banach space and $1 \le p \le 2$. As usually X is of Rademacher type $p(q)$ if there exists $C \in \mathbb{R}$ such that

$$\left(\int_0^1 \| \sum_{k \le n} r_k(t)\, x_k \|^q \, dt \right)^{1/q} \le C \left(\sum_{k \le n} \|x_k\|^p \right)^{1/p}$$

for every $x_1, x_2, \dots, x_n \in X$. Here $r_k(t)$ are the Rademacher functions,

$$r_0(t) = 1, \quad r_i(t) = \operatorname{sign} 2^n \pi t, \qquad i = 1, 2, \dots.$$

The smallest constant C satisfying this inequality is called type constant $T^{(n)}_{p(q)}(X)$ of X or Khintchin constant (see for instance [4]).

The most interesting cases are $p = q = 2$, $q = p$ and $q = p'$.

Let us note that sometimes is more convenient to use the following definition for type constant

$$T^{(n)}_{p(q)} = \sup \Big\{ \frac{\left(\frac{1}{2^n} \sum_{\theta_k = \pm 1} \| \sum_{k \le n} \theta_k x_k \|^q \right)^{1/q}}{\left(\sum_{k \le n} \|x_k\|^p \right)^{1/p}} : \sum_{k \le n} \|x_k\| \ne 0 \Big\}.$$

Let us mention, some known properties of the Khintchin constant. For instance,

- $1 \le T^{(n)}_{p(q)}(X) \le n^{1-1/p}$;
- the function $T^{(n)}_{p(q)}(X)$ is increasing in n, p, q, and $T^{(n)}_{1(1)}(X) = 1$;
- the function $T^{(n)}_{p(q)}(X)$ is submultiplicative in n.

Note that $C^{(n)}_{NJ} = \left(T^{(n)}_{2(2)} \right)^2$, where $C^{(n)}_{NJ}$ is n-th Jordan-von Neumann constant, first defined by Kato, Takahashi and Hashimoto in [3]. In this paper some properties of $C^{(n)}_{NJ}$ are proved and some connection between the Banach space to have some type (namely $\sup_n T^{(n)}_{p(q)}(x) < \infty$) and behaviour of the $C^{(n)}_{NJ}$ are considered.

Let $w = (w_1, w_2, \dots, w_n)$ with $w_1 \geq w_2 \geq \cdots \geq w_n > 0$ and $n = 2, 3, \dots$. For $1 \leq q < \infty$, the $n-$dimensional Lorentz sequence space, $d^{(n)}(w, q)$, is $\mathbb{R}^n$ with norm

$$\|x\|_{w,q} = (w_1 x_1^{*q} + w_2 x_2^{*q} + \cdots + w_n x_n^{*q})^{1/q},$$

where $(x_1^*, x_2^*, \dots, x_n^*)$ is the non-increasing rearrangement of $(|x_1|, |x_2|, \dots, |x_n|)$; that is $(x_1^* \geq x_2^* \geq \cdots \geq x_n^*)$. In the case when $w_k = k^{q/p-1}$, $k = 1, 2, \dots, n$ and $1 \leq q < \infty$, we have the classical n-dimensional Lorentz sequence space $l_{p,q}^n$.

Let $W_n = w_1 + w_2 + \cdots + w_n$.

Lemma. *If* $1 \leq q < \infty$, *then*

$$\left(\frac{W_n}{n}\right)^{1/q} \|x\|_q \leq \|x\|_{w,q} \leq w_1^{1/q} \|x\|$$

for all $x \in \mathbb{R}^n$.

This is the fact mentioned in Remark 1 (see [1]).

Theorem. *The following estimate holds for* $1 \leq q < \infty$:

$$T_{p(s)}^{(n)}(l_{w,q}) \leq \left(\frac{n w_1}{w_1 + w_2 + \cdots + w_n}\right)^{1/q} T_{p(s)}^{(n)}(l_q).$$

Proof. For to prove the estimate for the Khintchin constant of the space $l_{w,q}$ by Khintchin constant of the space l_q we are going to use first the right hand side inequality and then the left hand side inequality from the Lemma.

$$\begin{aligned}
\left(\frac{1}{2^n} \sum_{\theta_k = \pm 1} \|\sum_{k=1}^n \theta_k x_k\|_{w,q}^s\right)^{1/s} &\leq \left(\frac{w_1^{s/q}}{2^n} \sum_{\theta_k = \pm 1} \|\sum_{k=1}^n \theta_k x_k\|_q^s\right)^{1/s} \\
&\leq w_1^{1/q} T_{p(s)}^{(n)}(l_q) \left(\sum_{k=1}^n \|x_k\|_q^p\right)^{1/p} \\
&\leq w_1^{1/q} T_{p(s)}^{(n)}(l_q) \left(\sum_{k=1}^n \|x_k\|_{w,q}^p\right)^{1/p} \left(\frac{w_1}{w_1 + w_2 + \cdots + w_n}\right)^{1/q}.
\end{aligned}$$

Theorem is proved.

Corollary.

$$C_{NJ}^n(l_{w,q}) \leq n^{2/q'-1} \left(\frac{n w_1}{W_n}\right)^{2/q}$$

for $1 \leq q \leq 2$.

Proof. If we put $p = s = 2$ and use the fact $C_{NJ}^n(l_q) = n^{2/q'-1}$ for $1 \leq q \leq 2$ we get

$$C_{NJ}^n(l_{w,q}) \leq \left(\frac{n w_1}{W_n}\right)^{2/q} C_{NJ}^n(l_q) = \left(\frac{n w_1}{W_n}\right)^{2/q} n^{2/q'-1}.$$

Consider the generalized Clarkson inequality

$$\left(\frac{1}{2^n} \sum_{\theta_k = \pm 1} \|\sum_{k=1}^n \theta_k x_k\|^s\right)^{1/s} \leq n^{1/t-1/p} \left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p}$$

where $t = \min(2,p)$ for $s \le 2$ and $t = \min(s',p)$ for $s > 2$ (see for instance [5] and [2]). The inequality is proved under the condition $p \le q \le s$.

When speaking about type, the interesting case is $p \le 2$. So for $p \le q \le s \le 2$ we get $t = p$ and hence in this case the type constant $T^{(n)}_{p(s)}$ of the space l_q does not exceed 1, so it is equal to one. So we come to the inequality

$$T^{(n)}_{p(s)}(l_{w,q}) \le \left(\frac{nw_1}{w_1 + w_2 + \cdots + w_n}\right)^{1/q}.$$

Let now $q > 2$, i.e we are in the situation $p \le 2 \le q \le s$. Then $t = \min(s',p)$. When $p \le s'$ we came again to the same estimate of the type constant for $l_{q,w}$ as above by $\left(\frac{nw_1}{w_1+w_2+\cdots+w_n}\right)^{1/q}$. In the particular, but interesting case $s = q$ we get for the case $p > q' = s'$ that

$$\begin{aligned} T^{(n)}_{p(q)}(l_{w,q}) &\le n^{1/q'-1/p}\left(\frac{nw_1}{w_1 + w_2 + \cdots + w_n}\right)^{1/q} \\ &= n^{1-1/p}\left(\frac{w_1}{w_1 + w_2 + \cdots + w_n}\right)^{1/q} \\ &= n^{1/p'}\left(\frac{w_1}{w_1 + w_2 + \cdots + w_n}\right)^{1/q}. \end{aligned}$$

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ОЦЕНКИ ЗА КОНСТАНТАТА НА ХИНЧИН В ПРОСТРАНСТВАТА НА ЛОРЕНЦ ОТ РЕДИЦИ

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БЪЛГАРИЯ

Резюме: *В тази статия са получени някои оценки за константата на Хинчин, наричана още константа от тип-n за пространствата на Лоренц от редици. Като следствия са изведени оценки за различни случаи на параметрите. В частност, намерени са оценки за константата на Джордан-фон Нюман за пространствата* l_q.

Ключови думи: *константа на Хинчин, пространства на Лоренц от редици.*