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AN INVESTIGATION OF COMPLEX RIGID BODY MOTION

Svetoslav Nikolov[1,2], Valentin Nedev[1], Petar Kolev[1]

S.Nikolov@imbm.bas.bg

[1]*University of Transport, Geo Milev Str., 158, 1574 Sofia, BULGARIA*

[2]*Institute of Mechanics, Bulgarian Academy of Science, Acad. G. Bonchev Str., bl. 4, 1113 Sofia, BULGARIA*

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Abstract: *During recent years it has been increasing interest on the phenomena of chaos in gyroscopic systems. It is well-known that depending on the speed of rotation, a gyroscopic system may lose or gain stability. Despite the overwhelming number of studies reporting the occurrence of various chaotic structures, there is yet little known about construction details and generality of underlying bifurcation scenarios which gives rise to such chaotic (complex) behavior.*

Here, we report a detailed analytical and numerical investigation of the abundance of regular and chaotic behavior for rigid body (gyrostat) motion. The model contains 6 parameters that may be tuned to produce rich dynamical scenarios. Our results suggest that the heteroclinic structures with two, three, four and five fixed points from type saddle-focus occur.

1. Introduction

Modeling is a powerful tool in the simulation of processes in physics and technics dealing with different time and spatial scales, and in mechanical characterization of system parameters. It is also effective in the interpretation and design of experiments, as well as in the prediction of new effects and phenomena. It is clear that modeling would play an increasing role in improving our understanding of the physical processes in mechanical systems, under normal and abnormal conditions.

In an array of great discoveries in the twentieth century, three of them certainly belong to physics: 1) the theory of relativity of Albert Einstein (without ignoring the great merit of Henri Poincaré); 2) quantum mechanics, associated with a large number of scientists from different countries and 3) chaos theory, associated mostly with the name of the American theorist Edward Lorenz, a meteorologist. Again, Henri Poincaré has a contribution to its development, and later many scholars, the list of whose names cannot fit several dozen pages. It will be only mentioned that the first to use the word *chaos* in its modern understanding in science are Li and Yorke in 1975 [16], and that the authors of the concept of the strange attractor are Ruelle and Takens in 1971 [17].

There has been a large amount of recent interest in the investigation of gyroscope dynamics. The gyroscope has attributes of great utility to navigational and aeronautical engineering, biology, optics and et al. [7-11, 14]. Different types of gyroscope (with linear or nonlinear damping, fluid, et al.) are investigated for predicting the dynamic responses such as regular and chaotic motions [4, 12, 13].

Dissipative systems are a special class of dynamical systems. In general, dissipative mechanical systems in more than two dimensions have bounded and unbounded orbits depending on the energy. The bounded trajectories of these systems does not converge to an equilibrium point nor to a periodic or quasi-periodic orbit. In this case the flow is essentially aperiodic. A dynamical system is dissipative, if its phase volume contracts continuously, i.e. $D_i < 0 \left(i = 1, ..., n\right)$. Only dissipative dynamical systems have attractors.

Attractors are the adequate mathematical (geometrical) representations of time order and chaos that can be: stable equilibria, stable periodic motions (auto-waves) or strange attractors. Mathematical representations of spatial order and chaos are saddle equilibria, saddle periodic movements or complex saddle invariant sets.

It is well-known that a heteroclinic cycle is a sequence of trajectories connecting a set of fixed points in a topological circle. A classical heteroclinic cycle is a loop that consist of saddle equilibrium states connected to one another by their separatrices. The special case of a cycle consisting of one trajectory and one fixed point is usually called a homoclinic trajectory [19, 20].

A homoclinic trajectory $f(x,t)$ (or a homoclinic (separatrix) loop) is such that the 'inset' to a fixed point of an attractor, x_0, is the same as the 'outset' from the same point. It obeys the rule that $f(x,t) \to x_0$ for $t \to \infty, t \to -\infty$ [21]. According to Peixoto's theorem [22, 23], homoclinic bifurcations are structurally unstable and are therefore destroyed by small perturbations. Consequently, they are more difficult to identify than local bifurcation, because knowledge of the global properties of the phase space trajectories is required. Around a saddle-focus equilibrium a systematic characterization of homoclinicity was provided by Shilnikov [24]. In this scenario, reinjection occurs along a well-defined vector associated with a real system eigenvalues, with ejection from the vicinity of the equilibrium subsequently effected on a spiral path located on a transverse plane. A necessary condition for this mechanism is that the saddle-focus index $\delta = \left|\mathrm{Re}\left(\chi_2 / \chi_1\right)\right| < 1$, where χ_1 and χ_2 are the leading eigenvalues. Here we note that χ_1 determining the rate of approaching and χ_2 determining the rate of leaving the stable point. If thus Shilnikov condition is satisfied, an infinite number of nonperiodic trajectories coexist in the vicinity of a homoclinic trajectory bi-asymptotic to the saddle-focus.

A nice example is the Lorenz system [18], which has an important historical relevance in the development of chaos theory. Now this system is considered as a paradigmatic example of a chaotic system [1]. If $\nabla \dot{x}_i$ take both positive and negative values then this system is adaptive. Rigid bodies are bodies that cannot deform and change their shape, but the can translate and rotate. In [2], Neimark and Landa suggested that the Lorenz system admit a purely mechanical model of two rigid bodies- a lifting and the lifted axi-symmetric rotors. The system has the following property: when the rotor moves relative to the lifting body then the distribution mass remains unaltered in the space. Thus, the tensor of the system is constant and such a system is called gyrostat (see Figure 1). Further, it is assumed that its center of mass is fixed, and that the ellipsoid of inertia has the form of a rotation ellipsoid.

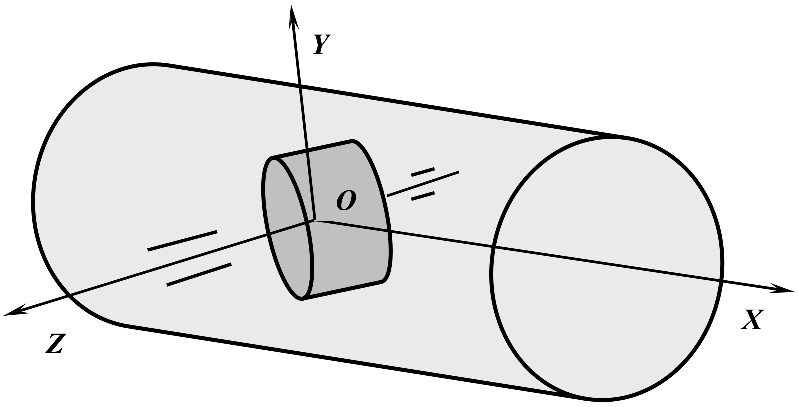

Figure 1. Two bodies - a lifting and lifted axi-symmetric rotor.

In case of small angular velocities, the motion of the system can be presented from the following differential equation [3,4]:

(1) $$\dot{M} = M \times AM + BM,$$

where M is the vector of kinematical moment about the coordinate system of the rigid body, $A = I^{-1} = diag(a_1, a_2, a_3)$ (where I is the inertial tensor) and B is constant matrix. For dissipative case, i.e. $div < 0$, the condition $TrB < 0$ is valid. In [5] for specific values of system parameters it is shown that two attractors have place. It is well known that system (1) has two particular cases: (i) those of Greenhill and (ii) Klein & Sommerfeld [6]. In these two cases the trajectories lie in integral surfaces. For some complicated cases, system (1) has two strange attractors.

According to Figure 1, the system (1) can be written in the form

(2) $$\begin{aligned} \dot{M}_1 &= c_1 M_1 + M_2 + c_2 M_2 M_3, \\ \dot{M}_2 &= -M_1 - c_3 M_2 + c_4 M_1 M_3, \\ \dot{M}_3 &= c_5 M_3 - c_6 M_1 M_2. \end{aligned}$$

In general, the investigation of such a model would start with the computation of steady states. The equilibrium (steady state) points of the system (2) are found by equating the right-hand sides of (2) to zero. Thus, it is easy to see that equilibrium points of the system (2) are

(3) **O₁:** $\bar{M}_1 = \bar{M}_2 = \bar{M}_3 = 0,$ *first fixed point*

(4) **O$_{2,3,4,5}$:** $$\bar{M}_1 = \frac{1}{c_1}\bar{M}_2\left(1 + \frac{c_2 c_6}{c_1 c_5}\frac{\bar{M}_2^2}{1 - \frac{c_2 c_6}{c_1 c_5 \bar{M}_2^2}}\right), \quad \bar{M}_3 = \frac{c_6}{c_1 c_5}\frac{\bar{M}_2^2}{1 - \frac{c_2 c_6}{c_1 c_5}\bar{M}_2^2},$$ *second, third, fourth and fifth fixed points*

$$\overline{M}_2^4 - \frac{1}{c_2^2 c_3 c_6}\left[c_5(c_2 + c_4) + 2c_1 c_2 c_3 c_5\right], \quad \overline{M}_2^2 + \frac{c_1 c_5^2}{c_2^2 c_3 c_6^2}(1 + c_1 c_3) = 0.$$

Because $(c_2 + c_4)^2 + 4c_1 c_2 c_3 (c_1 c_2 c_3 + c_4) > 0$, then the system (2) has fifth real fixed points- see Figure 2a.

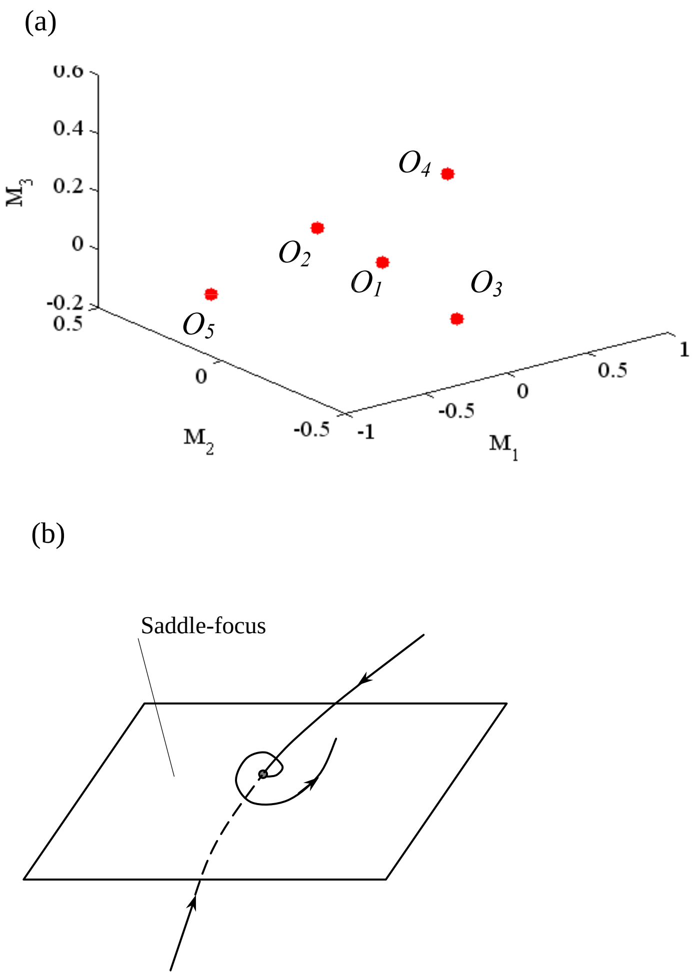

Figure 2. a: Fixed points of system (2) in $M_1M_2M_3$ coordinates; **b:** fixed point from type saddle-focus with unstable focus (complex eigenvalues with positive real part).

The divergence of the flow (2) is

(5) $$D_3 = \frac{\partial \dot{M}_1}{\partial M_1} + \frac{\partial \dot{M}_2}{\partial M_2} + \frac{\partial \dot{M}_3}{\partial M_3} = c_5 - (c_1 + c_3).$$

The system (2) is dissipative, when $D_3 < 0$, i.e. $c_5 < c_1 + c_3$. For example, in 1981 Leipnik and Newton [5] found that for $c_5 < 0.8$ the system is dissipative and all volumes in the phase space must contract uniformly even though cross-sectional and all trajectories except those trapped at the rest points diverge to infinity. Simulations suggest that for $c_5 = 0.175$ the system (2) has two strange attractors.

The paper is organized as follows: in Section 2 and 3 we present analytical and numerical results concerning the system (2) for different values of bifurcation (control) parameters c_1 and c_5. In Section 4 we discuss and summarize our results.

2. Qualitative analysis

In this section, we investigate the system (2), which presents an autonomous nonlinear 3D dynamical model.

Generally, in order to determine the character of fixed points (Eqs. (3) and (4)) we make the following substitutions into (2)

(6) $$M_1 = \bar{M}_1 + w_1, \quad M_2 = \bar{M}_2 + w_2, \quad M_3 = \bar{M}_3 + w_3.$$

Hence, after some transformations the system (2) has the form

(7) $$\begin{aligned} \dot{w}_1 &= -c_1 w_1 + b_1 w_2 + b_2 w_3 + c_2 w_2 w_3, \\ w_2 &= -b_3 w_1 - c_3 w_2 + b_4 w_3 + c_4 w_1 w_3, \\ \dot{w}_3 &= -b_5 w_1 - b_6 w_2 + c_5 w_3 - c_6 w_1 w_2, \end{aligned}$$

where

(8) $$\begin{aligned} &b_1 = 1 + c_2 \bar{M}_3, \quad b_2 = c_2 \bar{M}_2, \quad b_3 = 1 - c_4 \bar{M}_3, \\ &b_4 = c_4 \bar{M}_1, \quad b_5 = c_6 \bar{M}_2, \quad b_6 = c_6 \bar{M}_1. \end{aligned}$$

According to [15], the Routh-Hurwitz conditions for stability of fixed points (3) and (4) can be written in the form

(9) $$\begin{aligned} &p = c_1 + c_3 - c_5 > 0, \\ &q = c_1(c_3 - c_5) - c_3 c_5 + b_1 b_3 + b_2 b_3 + b_4 b_6 > 0, \\ &r = -[c_5(c_1 c_3 + b_1 b_3) - b_5(b_1 b_4 + b_2 c_3) + b_6(b_2 b_3 + b_4 c_1)] > 0, \\ &R = pq - r = (c_3 - c_5)[c_1(c_1 + c_3 - c_5) - c_3 c_5 + b_4 b_6] + b_1 b_3 (c_1 + c_3) + \\ &\qquad + b_2 b_5 (c_1 - c_5) - b_1 b_4 b_5 + b_2 b_3 b_6 > 0. \end{aligned}$$

Here the notations p, q, r and R are taken from [15]. The characteristic equation of the system (7) (which is equivalent of system (2)) can be written as

(10) $$\chi^3 + p\chi^2 + q\chi + r = 0.$$

Here we note that, the five fixed points at bifurcation parameters c_1 and/or c_5 are always from type *saddle-focus* – negative real eigenvalue and complex eigenvalues with positive real part (unstable focus) (see Figure 2b). These fixed points can be included in heteroclinic structures with two, three, four and fifth equilibriums, where their invariant manifolds W^+ and W^- meeting each other in a most intricate manner- see Figure 3.

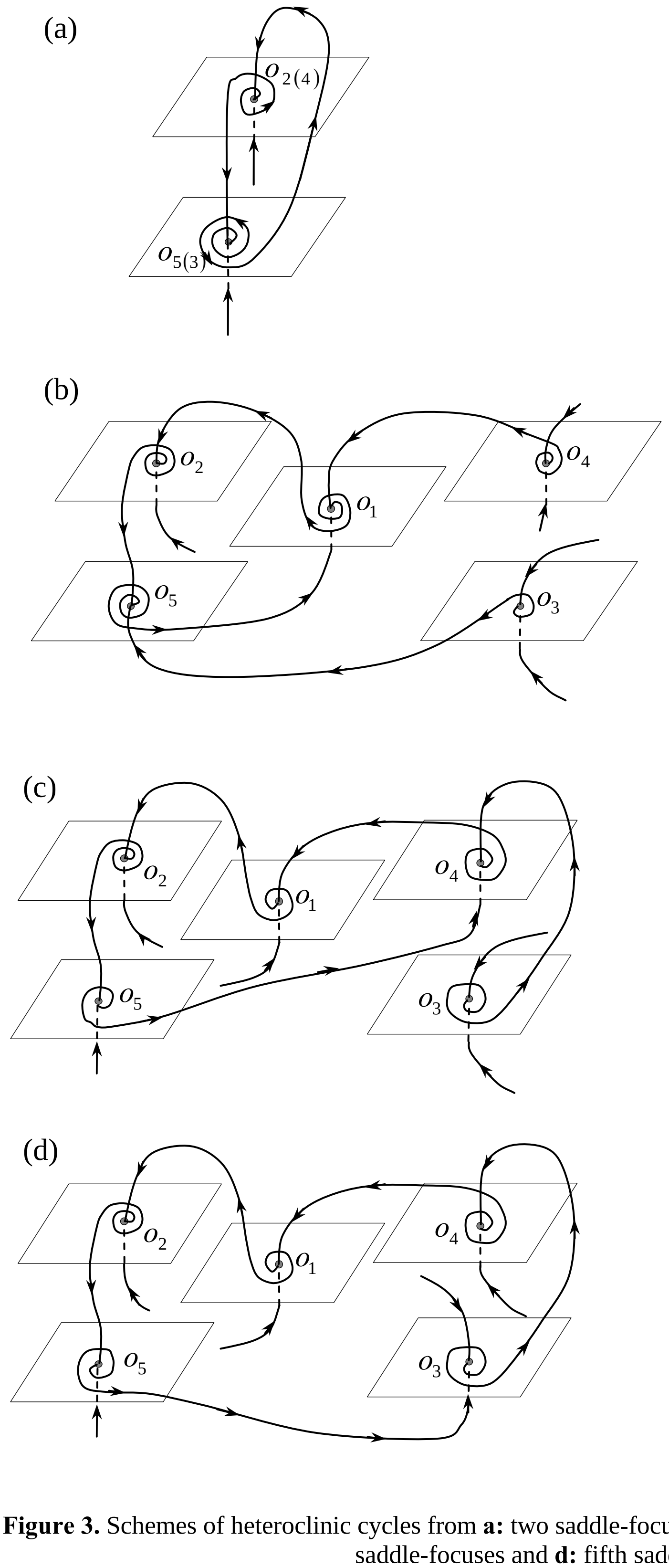

Figure 3. Schemes of heteroclinic cycles from **a:** two saddle-focus fixed points; **b:** three saddle-focuses; **c:** four saddle-focuses and **d:** fifth saddle-focuses.

A heteroclinic cycle is one of the common scenarios of the formation or death of a limit cycle when the limit cycle emanates from or approaches the heteroclinic cycle as a singular limit respectively [25]. There are known cases in which a unique limit cycle is born and certain criteria can be used to determine if this cycle must be stable or unstable. In our case here, the known results are not applicable, and we are forced to use numerical simulations and specific features of our system.

3. Numerical analysis

In the previous section, we obtained and shown some analytical results that we shall use in our numerical analysis in the system (2). According to [5, 26] the corresponding values of the dimensionless parameters $c_1 \div c_6$ are

(11) $$c_1 \in [0.4, 0.9], \quad c_2 = 10, \quad c_3 \in [0.4, 0.9], \quad c_4 = 5, \quad c_5 \in [0, 0.9], \quad c_6 = 5.$$

In order compare the predictions with numerical results, the governing equations of system (2), were solved numerically using MATLAB [27]. The initial conditions for all simulations are $(0.349, 0, -0.16)$ or $(0.349, 0, -0.18)$.

Figure 4 shows the bifurcation diagram for system (2): values of M_2 coordinate, $(M_2)_n$, are plotted against c_5 regarded as a continuously varying bifurcation (control) parameter. In this case the heteroclinic structure like to those in Figure 3b, 3c or 3d.We see that at $c_5 \in [0.175, 0.2]$the system (2) has chaotic solution. It is interesting to note that after $c_5 = 0.2$ (till the end of the interval) the inverse bifurcations occur and the system passes from chaotic regime to regular one. It is seen also that two symmetrical regular branches take place. We conclude also that an apparent sudden collapse in the size of a chaotic attractor occurs at a value of the control parameter $c_5 \approx 0.179$. Such a sudden qualitative change in a chaotic attractor is known as interior crisis.

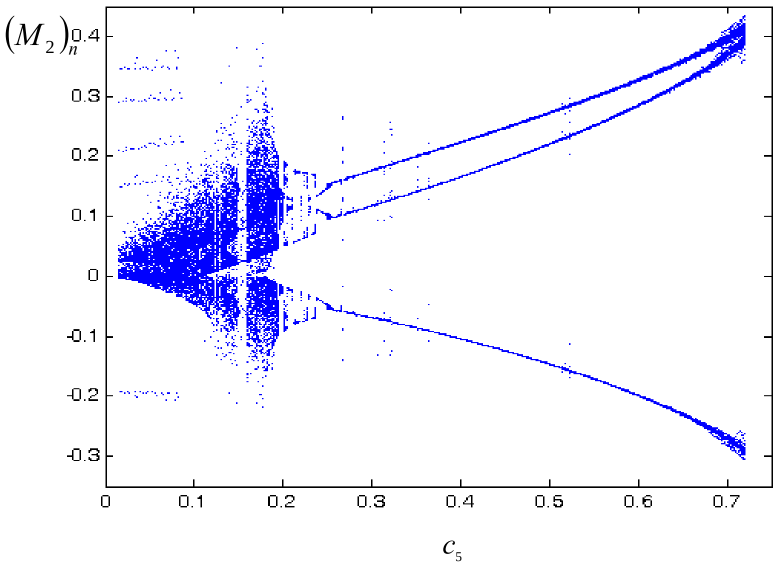

Figure 4. Bifurcation diagram $(M_2)_n$ versus c_5 generated by computer solutions of the system (2) computed with the parameters: $c_1 = c_3 = 0.4, c_2 = 10, c_4 = 5, c_5 \in [0.01, 0.78], c_6 = 5.$

In Figure 5, the bifurcation diagram of system (2) (as $c_1 \in [0.4, 0.88]$) is shown. It can be seen at $c_1 > 0.42$ chaotic solution with two strange attractors occurs. In this case heteroclinic structure from Figure 3a is valid.

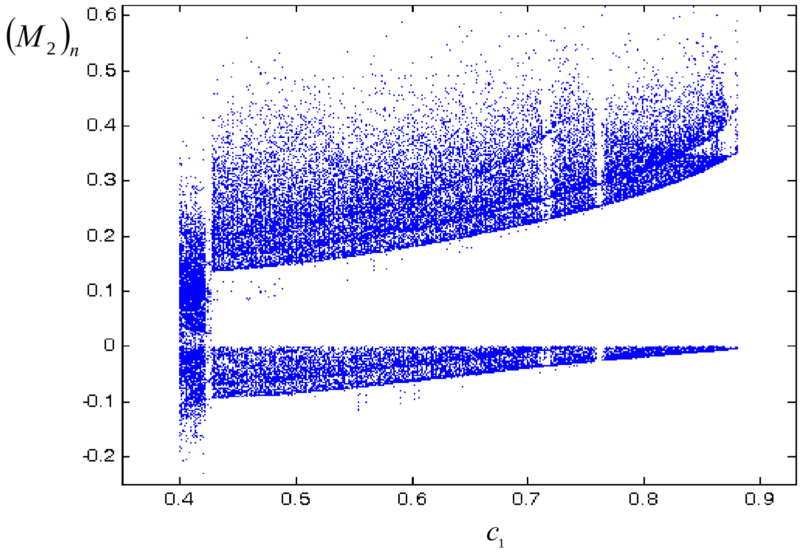

Figure 5. Bifurcation diagram $(M_2)_n$ versus c_1 generated by computer solutions of the system (2) computed with the parameters: $c_1 \in [0.4, 0.88], c_3 = 0.4, c_2 = 10, c_4 = 5, c_5 = 0.175, c_6 = 5.$

4. Summary and conclusions

An important feature of robust heteroclinic cycles is that they may attract nearby dynamics. What happens when a cycle loses stability? Such bifurcation may lead to the appearance of long period periodic orbits, other heteroclinic cycles, and more complicated dynamics.

The present paper studies how the dynamics and global behavior of system (2) vary, when we keep $c_2 = 10$, $c_3 \in [0.4, 0.9]$, $c_4 = 5$, $c_6 = 5$ and change c_1 and c_5. We focused our estimations on the bifurcation behavior, route to chaos and occurrence of heteroclinic structures (cycles). Our results suggest that the system (2) has (i) fifth unstable fixed points from type saddle-focus; (ii) heteroclinic structures including two, three, four and five fixed points. In the case when the system has two strange attractors, two heteroclinic structures including two fixed saddle-focuses take place.

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ИЗСЛЕДВАНЕ НА КОМПЛЕКСНОТО ДВИЖЕНИЕ НА ТВЪРДО ТЯЛО

Светослав Николов[1,2], Валентин Недев[1], Петър Колев[1]
S.Nikolov@imbm.bas.bg

[1]Висше Транспортно Училище "Т. Каблешков", ул. Г. Милев № 158, 1574 София,
[2]Институт по Механика – БАН, ул. Акад. Г. Бончев, бл. 4, 1113 София, БЪЛГАРИЯ

Ключови думи: *жиростат, хаотично поведение, числен анализ, качествен анализ*

Резюме: *През последните години в научната литература се забелязва засилен интерес към изучаване на появата на хаос в жироскопични системи. До сега е добре известно, че в зависимост от скоростта на въртене, една жироскопична система може да е в устойчиво или неустойчиво състояние. Въпреки огромния брой съществуващи научни публикации свързани с изследването на появата на различни хаотични структури, до сега много малко се знае за конструкционните детайли при появата на различни хаотични структури, както и за бифуркационните сценарии предизвикващи сложно (хаотично) поведение.*

В това наше изследване ние извършваме подробно аналитично и числено изучаване на възникването на регулярно и хаотично поведение на движението на твърдо тяло с една неподвижна точка (жиростат). Изследваният модел съдържа 6 константи, с варирането (изменението) на които може да се получат различни динамични сценарии. От нашите резултати се вижда, че се появяват хетероклинични структури с две, три, четири и пет фиксирани точки от вид седло-фокус.